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Meta-Logical Reasoning in Higher-Order Logic

Villadsen, Jørgen; Schlichtkrull, Anders; Hess, Andreas Viktor

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The terms and formulas of the FOL language are defined as the datatypes `tm` and `fm`, respectively. Variables are indexed using de Bruijn indices. The semantics of the language is defined using the function `sem` where `e` is the environment, i.e. a mapping of variables to elements of the universe `'u`, `f` maps constants to elements of `'u`, and `g` gives the semantics of the predicates. Most of the cases of `sem` should be self-explanatory, but the `Uni` case is complicated. The details are not important here, but it uses the universal quantifier (`!`) to consider all values of the universe `'u`. It also uses the lambda operator (`%`) to keep track of the indices of the variables.

Examples of Meta-Logical Reasoning

By using HOL as the meta-language for FOL it is possible to make use of proof assistants such as Isabelle to reason about certain properties of FOL. For example, let `syn` be a proof system for FOL implemented as a predicate in Isabelle of type `fm => bool` (an inductive definition). If it is sound then we can prove this in Isabelle as the theorem `syn fml ==> !e f g. sem e f g fml`. Likewise completeness (`!e f g. sem e f g fml`) `==> syn fml` can be proved if the proof system is indeed complete. For example, in Berghofer (2007) a natural deduction proof system for FOL is proven sound and complete. A tool for teaching logic based on natural deduction has recently been developed and proved sound by Villadsen (2015).

Meta-logical reasoning using the formalized semantics also enables formal proofs of sentences using only the semantics of the object language. For example, we wish to show that the sentence $\forall x. \forall y. P(y, x) \rightarrow P(y, x)$ is valid. We can prove this by first fixing two arbitrary elements u, w of the universe and show $P(y, x) \rightarrow P(y, x)$ for an arbitrary environment updated to map variable x to u and y to w, and then use the fact that any denotation g of predicates maps (P, u, w) to either true or false. In both cases $P(y, x) \rightarrow P(y, x)$ holds. This proof sketch can then be extended to a readable proof in Isabelle. The sentence can even be proved automatically in Isabelle using the formalized semantics:

```
theorem "!e f g. sem e f g (Uni (Uni (Imp (Pre ''P'' (Var 0) (Var 1))
                                          (Pre ''P'' (Var 0) (Var 1)))))"
```

```
by auto
```

On the other hand the sentence $\forall x. \forall y. P(x, y) \rightarrow P(y, x)$ is not valid. Therefore we can ask Isabelle to search for a counterexample, using the `nitpick` command, and it is able to find one.

Finally, the approach we have presented can even be used with HOL as the object language, that is, the semantics of HOL and a proof system for HOL can also be formalized in HOL, most extensively by Kumar (2014), although of course self-verification is not possible due to the second incompleteness theorem. Paulson (2014) has recently formalized the incompleteness theorems in Isabelle.

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