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Regularization of Piecewise Smooth Two-Folds

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1 Piecewise Smooth Systems

$X = (X^+, X^-)$ with vector-fields $(X^+, X^-)$ as a piecewise smooth (PWS) system. $\Sigma = \Sigma^+ \cap \Sigma^- : f(x, y, z) = 0$ is the switching manifold. Locally we take $f(x, y, z) > 0$ if (0) is divided into sliding $\Sigma_{x, y}$ and tangencies. $T$ see Fig. (a). On $\Sigma_{x, y}$ we adopt the Filippov convention [2] of sliding (see Fig. (b)) to obtain a vector-field $(X_{\Sigma}, \Sigma_{x, y})$.

2 Singularities

$p \in \Sigma$ is a singularity of $X^\pm$ with $X^\pm f(p) = 0$. A singularity is a fold if $X^\pm f(p) \neq 0$. Being visible when $\Sigma > 0$ (see Fig. (a)), invisible when $\Sigma < 0$ (see Fig. (c)). Here $X^\pm f = X^\pm f(x, y, z)$ is the Lie-derivative. A two-fold $p \in \Sigma$ is a fold from above and below: $X^\pm f(p) = X^\pm f(p) = 0$.

3 Two-Folds in $\mathbb{R}^3$

Proposition. [3] Generically, a two-fold $p$ in $\mathbb{R}^3$ is the transverse intersection of two lines $\Sigma$: $x = y = 0, z \in [-c^{-1}, c^{-1}]$. $\Gamma^\pm: y = 0, x \in [-c^{-1}, c^{-1}]$ consisting of fold points of $X^\pm$, respectively. The lines $\Gamma^\pm$ divide $\Sigma : y = 0$ into four separate regions:

- Stable sliding $\Sigma_{x, y}^+: x \leq 0, z \leq 0$
- Unstable sliding $\Sigma_{x, y}^-: x \geq 0, z \geq 0$
- Crossing downwards $\Sigma_{x, z}: x \leq 0, z \leq 0$
- Crossing upwards $\Sigma_{x, z}: x \geq 0, z \geq 0$

See Fig. (d)-(f). A two-fold is:

- Visible if $\Gamma^\pm$ are both visible (Fig. (d)).
- Visible-invisible if $\Gamma^\pm$ visible, $\Gamma^\mp$ invisible (Fig. (e)).
- Invisible if $\Gamma^\mp$ are both invisible (Fig. (f)).

Definition. A singular canard of a PWS is a trajectory of $\Sigma_{x, y}$ having a continuation through the two-fold singularity $p$.

The two-fold $p$ is an equilibrium of the vector-field $F_{\Sigma}X_{\Sigma}^+$ defined in $\Sigma_{x, y}^+ \cup \Sigma_{x, y}^- \cup \Sigma_{x, y}$ and with $F_{\Sigma} = H(x, z) \in \mathbb{R}$ a scalar smooth function which is positive (negative) for $x, z < 0$ ($x, z > 0$). Then:

- Proposition. [3] Non-degenerate singular canards exists if and only if $p$ corresponds to a node or a saddle of $F_{\Sigma}X_{\Sigma}^+$ and an eigenspace is contained in $\Sigma_{x, y} \cup \Sigma_{x, y}$.

See Fig. (g).

4 Regularization

- What happens to the two-fold/singular canards when we regularize the PWS system?
- Can we learn something about the PWS system by regularizing?

We consider the Sotomayor-Teixeira regularization [5]

$X_{\epsilon} = \frac{1}{2}X^+(1 + \phi(x, y, z)) + \frac{1}{2}X^-(1 - \phi(x, y, z))$, with $\epsilon < 1$ (see Fig. (h) and (i)). Writing $y = \epsilon y$ we obtain a hidden slow-fast system with $X_{\Sigma} = \epsilon X_{\Sigma}$ slow and $\hat{y}$ fast.

Theorem. [3] $X_{\epsilon}$ has critical manifolds: $\Sigma_0 = \Sigma_1$ (attracting), $\Sigma_1 = \Sigma_0^+$ (repelling) and a non-hyperbolic line $p: x = 0, y \in (-1, 1)$ (see Fig. (j)). On $\Sigma_{x, y}$ Reduced system = Filippov sliding system.

Note that in terms of $y = \epsilon y$ we have $\hat{p} = p$.

5 Blowup

To study the persistence of canards we blowup the nonhyperbolic line $p: x = \epsilon x, z = \epsilon^2, \epsilon = \epsilon^r, \epsilon > 0$ following the formulation of Krupa and Szmolyan [4]. We study the phase space using directional charts $\kappa_x : x = -1$, $\kappa_y : y = 1$ and a rescaling chart: $\kappa_y : \lambda = 1$. We obtain:

Theorem. [3] Singular canards $\Rightarrow$ (Primary, maximal) Canards as transverse intersections of continuations of Filippov slow manifolds $S_{\Sigma^+}$ and $S_{\Sigma^-}$ provided a certain non-resonance condition holds true. These maximal canards are $O(\sqrt{\epsilon})$ close to the singular canards.

Result and approach very similar to [6, 7] for folds in slow-fast systems in $\mathbb{R}^3$. But the geometry is very different.

6 Visible-Invisible Two-Fold

The two-fold is associated with forward and backwards non-uniqueness. By regularizing we can pick the “right orbits”.

Theorem. Consider the visible-invisible case and suppose as in Fig. (k) that there exists a singular cycle $\gamma_0$ satisfying certain non-degeneracy conditions, see also [1]). Then for $\epsilon \ll 1$ sufficiently small $X_{\epsilon}$ possesses an attracting limit cycle $\gamma_0$, satisfying $\gamma_0 = \gamma_0 + O(\sqrt{\epsilon})$.

PWS orbit $\gamma_0$ is therefore distinguished, as $\gamma_0 = \lim_{\epsilon \to 0+} \gamma_0$, among all the orbits through $p$. Note that these results hold true for all monotone regularization functions.

References