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Simulations of a single vortex ring using an unbounded, regularized particle-mesh based vortex method

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In recent work we have developed a new FFT based Poisson solver, which uses regularized Greens functions to obtain arbitrary high order convergence to the unbounded Poisson equation. The high order Poisson solver has been implemented in an unbounded particle-mesh based vortex method which uses a re-meshing of the vortex particles to ensure the convergence of the method. Furthermore, we use a re-projection of the vorticity field to include the constraint of a divergence-free stream function which is essential for the underlying Helmholtz decomposition and ensures a divergence free vorticity field. The high order, unbounded particle-mesh based vortex method is used to simulate the instability, transition to turbulence and eventual destruction of a single vortex ring. From the simulation data a novel method on analyzing the dynamics of the enstrophy is presented based on the alignment of the vorticity vector with the principal axis of the strain rate tensor. We find that the dynamics of the enstrophy density is dominated by the local flow deformation and axis of rotation, which is used to infer some concrete tendencies related to the topology of the vorticity field.

Key Words: High order vortex method, single vortex ring simulation, enstrophy, alignment of vorticity and strain rate tensor

1. Introduction

1.1. Vortex methods

Vortex methods is a class of numerical methods that predates the programmable computer. Vortex methods started out being quite popular due to its compact nature and limited number of degrees of freedom. As vorticity is a material property that is transported by the flow, it is conveniently solved in an Lagrangian formulation by using computational elements referred to as particles. The Lagrangian formulation additionally has the fortunate consequence in 2D simulations of eliminating the non-linear terms of the vorticity equation and thus reducing it to a simple diffusion equation.

However, as the years past, the computational power increased and the field of scientific computing matured, leading to an ever growing demand for accuracy in computation simulations. This caused the majority of the computational fluid dynamics society to focus on mesh based methods which has formal theory of convergence which can not be generalized to particle methods. Furthermore, as the number of computational elements increased drastically the particle based vortex method suffered from low computational efficiency especially in 3D simulations.

In recent years vortex methods have been developed towards a hybrid particle-mesh formulation in order to ensure the convergence of the method. Here the vorticity of the particles is interpolated to an auxiliary equispaced mesh where the velocity-vorticity coupling constituted by a Poisson equation is efficiently solved. The velocity is then interpolated back to the particles which in turn is frequently re-initiated with uniform form spacing (re-meshing) [1]. Re-meshing has been shown to be an important factor for the convergence of particle based methods [2].

Another challenge that one faces using particle based vortex methods for 3D simulations, is to preserve a divergence-free vorticity field. As the trajectory of the particles are solved numerically to a finite order of accuracy the solution is subject to an error accordingly. As a consequence the particles are formally no longer material points and the resulting error is manifested as a divergence in the vorticity field.

In resent work we have developed a particle-mesh based vortex method to perform simulations of unbounded flows. The method is implemented using particle re-meshing to ensure a numerical convergence and a re-projection which ensures a divergence-free vorticity field [2]. The high order method is achieved by using a recently developed unbounded Poisson solver [3] based on a regularization of the Greens function to obtain arbitrary high order convergence. Using the unbounded solver we are able minimize the size of the auxiliary mesh while imposing free-space boundary conditions. In this way we avoid introducing artifact in the simulation caused by approximate boundary conditions.

1.2. Simulation of a single vortex ring

Vortex rings have been subject to intense experimental and numerical research [4, 5, 6] in order to gain valuable insights into the evolution, instability and break down of vortex structures. Studying the flow topology of this fundamental
flow structure has extended the knowledge of dynamical instability and the eventual transition to turbulence of vortical flow structures. Although extensive, much of the analysis of the previous investigations has been focused on the topology of the vortex core structure and the time history of different moments of the flow field. Only little effort has been put into analyzing the local deformation of vorticity through the velocity gradient tensor governing the flow deformation.

In this work we present simulations of a single vortex ring using the aforementioned particle-mesh based vortex method with free-space boundary conditions. We present an extensive analysis on the alignment of the vorticity and the principal axes of the strain rate tensor which is shown to have a great significance to the changes of enstrophy density in the flow.

1.3. The alignment of the vorticity and the strain rate tensor and the significance of enstrophy

The vortex dynamics and enstrophy has been identified as an essential process in the characteristic energy cascade of turbulence. As the topology of the vorticity field is deformed into sheet- and tube-like structures, the conservation of angular momentum constitutes an energy transfer between different length scales of the flow. The coupling of the vorticity and kinetic energy is given by the enstrophy density which can be viewed as a quantity which gives the local rotational kinetic energy of the fluid. Thus the alignment of the vorticity vector is directly related to the isotropic properties of the turbulent energy transfer and has a significant role in many other topics of fluid dynamics. An example of this is found in aeroacoustics where the generated vortex sound is significantly reduced if the local vorticity becomes aligned with the velocity [7]. Thus an in depth knowledge of vortex deformation and alignment is essential for multiple disciplines in fluid dynamics.

2. Methodology of the proposed high order regularized vortex method

2.1. The governing equations

The velocity-vorticity formulation of the Navier-Stokes equations can be solved in a Lagrangian frame of reference by advecting computational particles which represents an elementary distribution of vorticity often referred to as a vortex blob. Hence the particle position \( x_p \) is solved in the Eulerian frame of reference by:

\[
\frac{d}{dt} x_p = v(x_p)
\]

where \( v \) is the velocity vector in the Eulerian frame of reference. In order to represent the vortex blob the particles are assigned a circulation \( \zeta \), which may be obtained by interpolating the circulation field \( \zeta(x) \) calculated by a regularization of the vorticity field \( \omega \) by:

\[
\zeta(x) = h^3 \int_{V_h} \omega(y) \Phi_m(x - y) dy
\]

Here \( \Phi_m \) is a moment conserving regularization function of order \( m \) and \( h \) is the cell length of the auxiliary mesh. In this work we construct the regularization function by de-convolving the Gaussian filter with an \( m \)-th order Taylor approximation. In this way the obtained method inherently resembles the de-convolution methods used in large eddy simulations (LES) [8].

The evolution of the particle vorticity is determined from the vorticity equation:

\[
\frac{D}{Dt} \omega = \nabla \cdot \nu + \nu \nabla^2 \omega \quad \text{where} \quad \omega \equiv \nabla \times v
\]

The regularized equation for the circulation is found by filtering the vorticity equation accordingly to Eq. (2) which leaves:

\[
\frac{D}{Dt} \zeta = \zeta \cdot \nabla \nu + \nu \nabla \zeta + O(h^m)
\]

Here \( v \) is the kinematic viscosity and an error term is included instead of the sub-grid stresses usually seen in LES equations. The replacement has been made under the assumption that all flow scales are properly resolved by the discretization after which the additional term represents a smoothing error of the regularization function and is thus in the order \( h^m \).

As the particles represents an integral of the vorticity we may reconstruct the vorticity field \( \omega(x) \) by utilizing the super-position principle which for \( N \) particles is:

\[
\omega(x) = \frac{1}{h} \sum_{p} N \xi_p W \left( \frac{x_p - x}{h} \right)
\]

Here \( W \) is an interpolation kernel and \( h \) is the cell length of the mesh.

In order to calculate the velocity field we use a functional orthogonal decomposition of the velocity field leaving:

\[
\nabla^2 v = \nabla \left( \nabla \cdot v \right) - \nabla \times \left( \nabla \times v \right)
\]

For an incompressible flow the velocity field is divergence free which can easily be obtained by nullifying the former term on the right-hand-side of Eq. (6) and thus by defining the vorticity as the curl of the velocity field one obtains:

\[
\nabla \times v = -\nabla \times \omega
\]

This equation is recognized as a Poisson equation and can thus be solved for unbounded boundary conditions to an arbitrary high order by a regularization Green's function solution [2].

2.2. Correction of divergence in the vorticity field

From the definition of vorticity in Eq. (3) we see that the
vorticity field is per definition divergence-free as:
\[ \nabla \cdot \omega = \nabla \cdot (\nabla \times \nu) = 0 \quad (8) \]
The velocity field may be described by a Helmholtz decomposition
\[ \nu = \nabla \varphi + \nabla \times \psi \quad (9) \]
where \( \varphi \) is the velocity potential and \( \psi \) is the stream function. The Helmholtz decomposition replaces the three dimensional velocity vector with four components (one from the velocity potential and three from the stream function) by which we have to enforce the constraint of a solenoidal stream function
\[ \nabla \cdot \psi = 0 \quad (10) \]
From Eqs. (3) and (9) we may write the vorticity as:
\[ \omega = \nabla \times \nabla \times \psi = \nabla (\nabla \cdot \psi) - \nabla \cdot \nabla \times \psi \quad (11) \]
which due to the constraint of the Helmholtz decomposition becomes:
\[ \omega_\omega = - \nabla \cdot \psi \quad (12) \]
It is now seen that the divergence of the vorticity field is:
\[ \nabla \cdot \omega_\omega = - \nabla \cdot (\nabla \cdot \psi) \quad (13) \]
In theory, the right-hand-side is zero due to the restriction of the Helmholtz decomposition (cf. Eq.(10)). However, the trajectory of the vortex particles, which essentially is represented by the stream function, is only solved numerically to a finite order of accuracy. As mentioned earlier in Sec. 1.1, this gives cause to an error which, as seen in Eq. (13), manifests itself in a divergence in the vorticity field. Hence the vorticity field should be corrected accordingly to
\[ \omega = \omega_\omega + \nabla (\nabla \cdot \psi) \quad (14) \]
where the divergence of the stream function is found by solving the Poisson equation of Eq. (13).

2.3. Analysis of the enstrophy by the alignment of the vorticity and the strain rate tensor
In the following section an inviscid assumption is made in order to exclusively investigate the dynamics of the enstrophy. From the vorticity equation (Eq. (3)) it is seen that the only non-diffusive term is the stretching term which may be re-written by decomposing the velocity gradient tensor into the strain rate tensor \( S = (\text{grad}(v) + \text{grad}(v^2))/2 \) and rotation rate tensor \( R = (\text{grad}(v) - \text{grad}(v^2))/2 \) as:
\[ \omega \cdot \nabla v = \omega \cdot (S + R) = \omega \cdot S \quad (15) \]

It is here noted that the vortex stretching is given by the dot product of the vorticity and the strain rate tensor. Given a sufficiently small control volume the vorticity can be viewed as the angular momentum of the volume per moment of inertia \cite{9}. Furthermore, the strain rate tensor may be regarded as the change of the moment of inertia which, due to the conservation of angular momentum, causes a change in vorticity if properly aligned with the vorticity. In a similar fashion we will extend this view by interpreting the enstrophy density (defined by \( \varepsilon = (\omega \cdot (\omega \cdot S))/2 \)) as the rotational kinetic energy per moment of inertia. By substituting Eq. (15) into Eq. (3) and applying the dot product with the vorticity vector, an equation for the material derivative of the enstrophy is derived for the inviscid case (i.e. \( v = 0 \)):
\[ \frac{D}{Dt} \varepsilon = \omega \cdot (\omega \cdot S) \quad (16) \]
Described in the principle axes of the strain rate tensor (denoted henceforth by a tilde) the strain rate tensor is diagonalized containing only its eigenvalues which represents the strain rate in the principal directions. Hence the enstrophy equation may be written using the eigenvalues of the strain rate tensor:
\[ \frac{D}{Dt} \varepsilon = \tilde{\omega} \tilde{\omega} \tilde{\lambda} + \tilde{\omega} \tilde{\lambda} + \tilde{\lambda}^2 \quad (17) \]
where the eigenvalues are ordered as \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \) and
\[ \lambda_1 + \lambda_2 + \lambda_3 = \nabla \cdot \nu = 0 \quad (18) \]
due to the incompressibility criteria. Hence we may write the three contributions in Eq. (17) using the first term as an example
\[ \tilde{\omega} \tilde{\lambda} = - \tilde{\omega} \tilde{\lambda} \lambda_1 \quad (19) \]
From this we can deduce that the dynamical contributions to the change in enstrophy consists of the fluid deformation in the plane perpendicular to the axis of rotation. Furthermore, the essential parameters of the change in enstrophy may be summarized as 1) the alignment of the vorticity vector with the principal axes of the strain rate tensor and 2) the relation of the eigenvalues of the strain rate tensor. Where the former may be described by the normalized dot product of the vorticity and the eigenvectors the latter is represented by the normalized skewness of the eigenvalues referred to as the deformation mode:
\[ \text{mode} (S) = \sqrt{2} \text{skewness} (\lambda_1, \lambda_2, \lambda_3) \quad (20) \]
Here a deformation mode of -1 corresponds to a disc-like deformation whereas a deformation mode of 1 corresponds to a tube-like deformation of a initially spherical fluid element.
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Fig. 1. The magnitude of the vorticity at five different time stages of a single vortex ring simulated at a circulation based Reynolds number of 10,000. The red and blue color indicate regions of high and low vorticity, respectively. A) $t \Gamma/D^2 = 15.3$ The initial vortex ring. B) $t \Gamma/D^2 = 31.1$ The vortex ring becomes unstable. C) $t \Gamma/D^2 = 39.9$ Secondary structures appear around the core. D) $t \Gamma/D^2 = 46.7$ The vortex core becomes highly irregular. E) $t \Gamma/D^2 = 55.6$ The vortex ring ruptures into turbulence. F) $t \Gamma/D^2 = 66.4$ The turbulence decays accordingly to viscous effects.

Fig. 2. Analysis of the three components contributing to the material derivative of the enstrophy in Eq. (17) related to each of the principal axes of the strain rate tensor for stage B shown in Fig. 1. The red iso-surfaces represents a positive change of enstrophy density, the blue iso-surfaces represents a negative change of enstrophy density, and the white iso-surfaces represents the vortex ring core which is used as a spatial reference.
### Probable types of deformation and vorticity alignment

<table>
<thead>
<tr>
<th>Stage A</th>
<th>Stage B</th>
<th>Stage C</th>
<th>Stage D</th>
<th>Stage E</th>
<th>Stage F</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Alignment with the 1st principal axis" /></td>
<td><img src="image2" alt="Alignment with the 2nd principal axis" /></td>
<td><img src="image3" alt="Alignment with the 3rd principal axis" /></td>
<td><img src="image4" alt="Alignment with the 1st principal axis" /></td>
<td><img src="image5" alt="Alignment with the 2nd principal axis" /></td>
<td><img src="image6" alt="Alignment with the 3rd principal axis" /></td>
</tr>
</tbody>
</table>

![Fig. 3. Investigation of the deformation of the vorticity field for the six stages shown in Fig. 1. Surface plots: Alignment of vorticity with the principal axis of the strain rate tensor for the different deformation modes. A negative deformation mode corresponds to a disc deformation whereas a positive deformation mode corresponds to a tube deformation. The right most column shows the types of vorticity deformation which is present at the given stages. Here the red axis indicates the alignment of the vorticity vector.](image7)
3. Results

3.1. Simulations of a single vortex ring at Re$_r$ = 10,000

Using the vortex method summarized above we simulate the evolution of a single vortex ring using the following parameters for Reynolds number, time step size and mesh cell size of:

$$\text{Re}_r = \frac{\Gamma}{\nu} = 10^4, \quad \delta t = \frac{0.1}{\max(|\omega|)}, \quad \delta x = \frac{D}{128}$$

Here $\Gamma$ is the circulation of the vortex ring cross-section and $D$ is the vortex ring diameter. In the presented implementation we use numerical schemes which is a 3$^{rd}$ order scheme for interpolation between particles and mesh, a 3$^{rd}$ order scheme for calculating the trajectory of the particles, and a 4$^{th}$ order finite difference scheme for calculating spatial derivatives. In Fig. 1 the magnitude of the vorticity is presented at six stages of the simulation. The vortex ring is initiated as a perturbed Gaussian distribution which becomes unstable creating secondary vortex structures. At some point, the instability destabilizes the vortex ring leading to a destruction of the vortex core into a turbulent field which eventually decays and dies out.

3.2. Analysis of the enstrophy by the alignment of vorticity and the deformation mode of the strain rate tensor

Using the analysis of Sec. 2.3 based on the eigenvalues of the strain rate tensor the components contributing to the material derivative of the enstrophy (cf. Eq. (17)) is investigated by a field analysis in Fig. 2 for stage B (cf. Fig. 1). It is here seen that clear regions are dominated by different components representing different consequences of the flow deformation. It is seen that the vortex core is dominated by the component related to the second eigenvalue whereas the secondary vortical structures are dominated by the components related to the first and third eigenvalue of the strain rate tensor.

This tendency is also shown in Fig. 3 where the probability of the vorticity alignment and deformation mode of the strain rate tensor are shown for the six different stages. It is seen that in the early stages the clear majority of the vorticity is aligned with the principal axis related to the intermediate eigenvalue $\lambda_2$ with both disk and tube deformation. As the vortex ring becomes unstable and secondary vortex structures are formed a larger part of the vorticity is aligned with the first and third eigenvalues. The correlation of the vorticity alignment with the first and third eigenvectors and the secondary vortex structures corresponds to what is shown in Fig. 2.

In the late stages where the topology of the vortex ring is disrupted it is seen that the vast majority of the vorticity is aligned with the principal axes which is related to either the first or the second eigenvalue and the flow deformation is solely that of a disk deformation. The results at this stage corresponds well with that reported by Ashurst [10] for the alignment of vorticity in free turbulence.

It is here emphasized that the presented analysis regards the dynamics of the local vorticity and enstrophy density and does not represent the macroscopic topology of the vorticity field.

4. Conclusion

A high order particle-mesh based vortex method has been presented for the simulation of vortical flows with free-space boundary conditions. The high order was achieved by a high order regularization of the vorticity field and a re-projection of the vorticity field applied to ensure a divergence-free vorticity field.

The presented vortex method was used to simulate the evolution, instability and break-down of a single vortex ring. By considering the angular momentum and rotational kinetic energy of an infinitesimal material volume of fluid a novel analysis of the vortex dynamics was made. The analysis showed that the different stages of the vortex ring break-down display clear differences in the dynamics of the enstrophy density. A field analysis of the dynamics of the enstrophy density furthermore showed different tendencies for primary and secondary vortex structures indicating a strong relation to the topology of the vortical structures.

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