Low-dimensional approximations for Finite Element Models of mechanical systems

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Summary. The present study is dedicated to the dimension reduction of high-dimensional FE models of mechanical systems in which low-dimensional behaviour is observable. A low-dimensional model of the such a FE model is constructed and the bifurcation diagram of the low-dimensional system is compared with that of the FE model in order to investigate the range of validity of the low-dimensional model.

Low-dimensional behaviour of high-dimensional mechanical models

We construct a low-dimensional approximation of a FE model of some mechanical system (e.g. Figure 1) and check how well the low-dimensional model captures the essential behaviour by comparing the bifurcation diagrams.
We do this because most mechanical systems are described by physical laws in the form of partial differential equations, and in many applications — industrial as well as academic — the systems are too complex to be handled analytically. When confronted with these problems where standard analytical methods are not suitable, usually one resorts to numerical approximations such as FE modeling.
Often numerical approximations have the drawback of being so high-dimensional that numerical bifurcation methods for obtaining the essential dynamical system behaviour are unlikely to be successful due to computational limitations.
This curse of high-dimensionality can be alleviated by dimension reduction methods. When macroscopic behaviour can be observed, it indicates the existence of an attracting low-dimensional submanifold of the phase space such as periodic orbits or tori.

The object of interest in this study is high-dimensional FE models of driven nonlinear mechanical systems where low-dimensional behaviour has been observed. The considered FE model will typically be some variant of eq. (1)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{g}(t) \qquad \text{where } \mathbf{x} \in \mathbb{R}^n,\ \mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}} \tag{1}$$

where n is large, **f** could be some nonlinear function of displacements and velocities, **x**, and **g** is a T-periodic function of time, t.

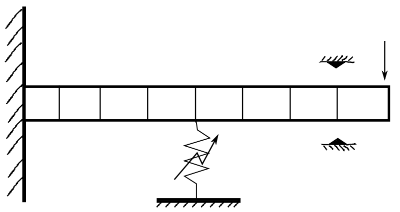

Figure 1: Sketch of a driven beam with a nonlinear spring and an impact.

In dynamical systems theory the low-dimensional manifolds are referred to under different names such as inertial manifolds, slow manifolds and center manifolds - all of which are conceptually very similar in the present subject. In the next section dimension reduction is introduced in terms of the slow manifold.

Mathematical Assumptions and Modelling

Consider the slow manifold approach for the n-dimensional ordinary differential equation (ODE)

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}, \mu) \qquad \forall(\mathbf{u}, \mu) \in \mathbb{R}^{\mathrm{n}\times\mathrm{k}}, \tag{2}$$

where F is an n-dimensional nonlinear function. Letting n be large this system could very well be the representation of some FE model of a mechanical system. Let us assume the existence of a spectral gap that seperates the slow and fast dynamics of eq. (2)

$$\dot{\mathbf{x}}_s = \mathbf{F}_s(\mathbf{x}_s, \mathbf{x}_f, \mu) \qquad \forall \mathbf{x}_s \in \mathbb{R}^{\mathrm{m}}, \tag{3}$$

$$\varepsilon\dot{\mathbf{x}}_f = \mathbf{F}_f(\mathbf{x}_s, \mathbf{x}_f, \mu) \qquad \forall \mathbf{x}_f \in \mathbb{R}^{\text{n-m}} \text{ and } 0 < \varepsilon \ll 1 \tag{4}$$

where n $\gg$ m and the subscripts s and f denote slow and fast, respectively, then the system might be restricted to a slow manifold. In general all of the aforementioned low-dimensional manifolds give rise to a slaving property — a map from the low-dimensional space to the high-dimensional space defined by

$$\Phi : \mathbf{x}_s \mapsto \mathbf{x}_f.$$

Given the existence of this slow manifold, $x_f = \Phi(x_s)$, eq. (3) can be made independent of the slaved variables, which leads to a low-dimensional dynamical system. The reduction by the slow manifold is given by

$$\dot{\mathbf{x}}_s = \mathbf{F}_s(\mathbf{x}_s, \Phi(\mathbf{x}_s)) \qquad \forall \mathbf{x}_s \in \mathbb{R}^{\mathrm{m}}. \tag{5}$$

This means that trajectories can be computed without error and bifurcation analysis might now become realistic. Notice that, although the dimensionality of the problem was reduced, the complexity of the problem was shifted to a large number of Cauchy problems that have to be solved in order to obtain the slaving relationship Φ. For the slow manifold it is *n-m* Cauchy problems that are derived by substituting the slaving $x_f = \Phi(x_s)$ into (4)

$$\varepsilon \frac{\partial \Phi}{\partial \mathbf{x}_s} \mathbf{F}_s(\mathbf{x}_s, \Phi(\mathbf{x}_s)) = \mathbf{F}_f(\mathbf{x}_s, \Phi(\mathbf{x}_s)), \tag{6}$$

which is *n-m* first-order nonlinear PDEs. In a first order approximation Φ can be captured as the solution of

$$0 = \mathbf{F}_f(\mathbf{x}_s, \Phi(\mathbf{x}_s)). \tag{7}$$

Next we consider a fundamental example in which the slow manifold is explicitly given. The example is illustrative of the strength of the idea but also acts as a warning because the dynamical system is only partly restricted to the slow manifold.

A classical example of a slow manifold

Consider the van der Pol equation, $\ddot{u} - \mu(1 - u^2)\dot{u} + u = 0$ ($\mu \gg 1$), by a change of variables the following slow-fast system can be achieved

$$\dot{x}_s = x_f \tag{8}$$

$$\varepsilon \dot{x}_f = -x_s + x_f - \frac{1}{3}x_f^3 \qquad \text{where} \quad 0 < \varepsilon = \frac{1}{\mu^2} \ll 1. \tag{9}$$

Considering the limit $\varepsilon \to 0$ the slow manifold is given by $0 = -x_s + x_f - \frac{1}{3}x_f^3$ and it is sketched in Figure 2 where thick gray line is the slow manifold and the black line is the low-dimensional manifold. It is evident that a naive slow manifold reduction is not a good choice for the dimension reduction in this particular example, it does however offer some crucial geometrical insight to the low-dimensional manifold through the fold points.

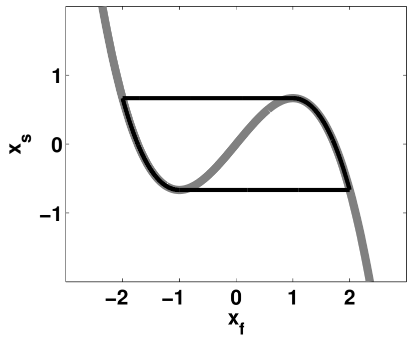

Figure 2: Sketch of the dynamics for the van der Pol equation.

Exploration, Aim & Method

The present study is an exploration of low-dimensional manifolds — e.g. center and/or slow manifolds — of FE models of mechanical systems and the aim is to check how well the low-dimensional models represent the essential behaviour of the high-dimensional full systems, i.e. to check the correspondence between the bifurcation diagrams.

As of now the work is still in progress. The mechanical system of the study is a FE model of a driven beam with a nonlinearity e.g. through friction and/or springs. The plan is to construct a low-dimensional system of the FE model by introducing a slaving like the one used by Jiang et al. [2], however, differing by the numerical construction of the low-dimensional manifold through explicitly including the bifurcation-parameters by continuation methods i.e. letting

$$\Phi : (\mathbf{x}_s, \mu) \mapsto \mathbf{x}_f, \tag{10}$$

and by considering a different nonlinearity.

The hope is that this analysis will reveal some limitations and in turn offer some insight on how to solve these.

References

[1] Rega G., Troger H. (2005) Dimension reduction of dynamical systems: methods, models, applications. *Nonlinear Dynamics* **41**:1-15

[2] Guckenhiemer J., Holmes P. (1983) Nonlinear Oscillations, dynamical systems, and bifurcations of vector fields. Springer Verlag.

[3] Jiang D., Pierre C., Shaw S.W. (2005) Nonlinear normal modes for vibratory systems under harmonic excitation. *J. Sound and Vib.* **288**:791-812