

HYBRID STATE-SPACE TIME INTEGRATION OF ROTATING BEAMS

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Summary. Modeling and efficient design of wind turbines require efficient and accurate computational methods for dynamic analysis of the different components. In the present paper an efficient hybrid formulation for beams in a rotating frame of reference is presented for analysis of the rotor system. It is demonstrated that the equations of motion take a particularly simple form when starting from a hybrid state-space in terms of local displacements relative to the rotating frame and absolute velocities using similar interpolation. The equations of motion are formulated for small finite deformation beam elements in terms of translational as well as rotational degrees of freedom and include the effect of geometric stiffness. The dynamic equations are derived from Lagrange's equation and combined with a kinematic relation into a convenient hybrid state-space format.

1 Hybrid state-space formulation

It has recently been demonstrated for translation based (solid) elements that the equations of motion take a particularly simple form when starting from a hybrid state-space in terms of local displacements \mathbf{u} relative to the rotating frame and absolute velocities \mathbf{v} [1]. In the present paper this approach is extended to small finite deformation beam elements with rotational degrees of freedom as well by introduction of an additive term to the kinematic relation.

Consider a structure located in a local frame rotating around its origin with local angular velocity vector $\boldsymbol{\Omega}^T = [\Omega_1, \Omega_2, \Omega_3]$. The local position of the structure is described in terms of the N nodes, which are collected in the array $\mathbf{x}^T = [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_N^T]$. The position vector \mathbf{x} may be represented as a sum of the initial position \mathbf{x}_0 and a displacement vector \mathbf{u}

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{u} \quad (1)$$

For 3D beam elements accommodating translational as well as rotational degrees of freedom the displacement vector is conveniently organized in the $2N$ block format $[\mathbf{u}_1^T, \boldsymbol{\varphi}_1^T, \mathbf{u}_2^T, \boldsymbol{\varphi}_2^T, \dots, \mathbf{u}_N^T, \boldsymbol{\varphi}_N^T]^T$. The absolute velocity of node j can then be formulated as

$$\mathbf{v}_j = \begin{bmatrix} \dot{\mathbf{u}}_j + \tilde{\boldsymbol{\Omega}}\mathbf{x}_j \\ \dot{\boldsymbol{\varphi}}_j + \boldsymbol{\Omega} \end{bmatrix}, \quad j = 1, 2, \dots, N \quad (2)$$

where $\tilde{\Omega} = \Omega \times$. The global velocity components are collected in the array $\mathbf{v}^T = [\mathbf{v}_1^T, \mathbf{v}_1^T, \dots, \mathbf{v}_N^T]$, thus the system format for the absolute nodal velocities can be obtained from (2) by utilization of the expression (1),

$$\mathbf{v} = \dot{\mathbf{u}} + \tilde{\Omega}_D \mathbf{u} + \tilde{\Omega}_D \mathbf{x}_0 + \Omega_C \quad (3)$$

The matrices $\tilde{\Omega}_D$ and Ω_C are introduced as

$$\tilde{\Omega}_D = \begin{bmatrix} \tilde{\Omega} & & \\ & \mathbf{0} & \\ & & \ddots \end{bmatrix}, \quad \Omega_C = \begin{bmatrix} \mathbf{0} \\ \Omega \\ \vdots \end{bmatrix} \quad (4)$$

where $[\tilde{\Omega} \ \mathbf{0}]$ and $[\mathbf{0}^T \ \Omega^T]^T$ are repeated for each node of the structure.

A key feature in the present formulation leading to particular simplifications of the discretized equations of motion is to use the same shape function representation for the local position \mathbf{x}_ξ as well as the absolute velocity \mathbf{v}_ξ in terms of the nodal components \mathbf{x} and \mathbf{v} , respectively. The interpolation for the position \mathbf{x}_ξ can be expressed as

$$\mathbf{x}_\xi = \mathbf{N}(\xi) \mathbf{x} = \mathbf{N}(\xi) [\mathbf{x}_0 + \mathbf{u}] \quad (5)$$

where $\mathbf{N}(\xi)$ denotes a suitable interpolation in terms of the normalized initial coordinate ξ . The interpolation of the absolute velocity is similarly expressed as

$$\mathbf{v}_\xi = \mathbf{N}(\xi) \mathbf{v} = \mathbf{N}(\xi) [(\partial_t + \tilde{\Omega}_D) \mathbf{x} + \Omega_C] \quad (6)$$

This particular choice of interpolation leads to a formulation where the angular velocity can be extracted from the integrals defining the inertia effects such as centrifugal and Coriolis forces for the individual elements. It should be noted that if a direct point-wise time differentiation were used, the combined time differentiation and convection term arising from the angular velocity of the rotating frame would appear to the left of the shape function matrix \mathbf{N} . As a consequence the matrices related to inertia effects in the discretized formulation need recalculation on element level for changing angular velocity. In the present formulation this change is accounted for by a simple pre- or post-multiplication of the system mass matrix with the angular velocity matrix $\tilde{\Omega}_D$ and the vector Ω_C .

Expressing the kinetic energy T in terms of the local mass matrix, \mathbf{M}

$$T = \frac{1}{2} \mathbf{v}^T \mathbf{M} \mathbf{v} \quad (7)$$

and the internal forces $\mathbf{g}(\mathbf{u})$ from an energy potential $G(\mathbf{u})$

$$\mathbf{g}(\mathbf{u}) = \frac{dG(\mathbf{u})}{d\mathbf{u}^T} \quad (8)$$

the equations of motion follow from Lagrange's equations:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\mathbf{u}}^T} \right) - \frac{\partial T}{\partial \mathbf{u}^T} + \frac{\partial G}{\partial \mathbf{u}^T} = \mathbf{f} \quad (9)$$

where \mathbf{f} is the nodal force array.

The equations of motion are conveniently expressed in a hybrid state-space format in terms of the local displacements \mathbf{u} and absolute velocities \mathbf{v} . The dynamic equations of motion follow from (9) in the form

$$\left(\partial_t + \tilde{\Omega}_D\right) \mathbf{M}\mathbf{v} + \mathbf{g}(\mathbf{u}) = \mathbf{f} \quad (10)$$

For the current use the mass matrix is assumed independent of time and the kinematic equation (3) and the dynamic equation (10) are combined into the following state space format:

$$\begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{v}} \end{bmatrix} + \begin{bmatrix} \mathbf{g}(\mathbf{u}) + \tilde{\Omega}_D \mathbf{M}\mathbf{v} \\ \mathbf{M}\tilde{\Omega}_D^T \mathbf{u} + \mathbf{M}\mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ -\mathbf{M} \left(\tilde{\Omega}_D^T \mathbf{x}_0 - \Omega_C \right) \end{bmatrix} \quad (11)$$

The state space format is hybrid in the sense that \mathbf{u} is the local displacements while \mathbf{v} is the absolute velocity. The discretized equations are separated into a computationally attractive symmetric and anti-symmetric block format. The inertial terms arising from the rotation of the local frame are solely represented by two gyroscopic terms and generalization to accelerated rotation is straight forward leading to a more general and computational attractive format compared to the classic block matrix format in terms of local components, [2].

Using a linearized formulation in terms of the constitutive and an initial stress based geometric stiffness matrix, the equations of motion can be solved using a very efficient two-step integration algorithm without the need for internal iterations. The algorithm is based on an integrated form of the equations of motions as proposed in [3]. The angular velocity is represented by its mean value over the time increment and the internal forces $\mathbf{g}(\mathbf{u})$ are represented in the form of end-point values supplemented by a term involving the increment of the geometric stiffness, [4].

2 Example: Transient acceleration of prismatic beam

The properties and accuracy of the hybrid state space algorithm are illustrated considering a spin-up sequence of a prismatic beam rotating about a fixed axis. The beam is originally introduced by [5], but parameters corresponding to [6] have been used. These are equivalent to a beam of length $L = 10$ m, with a square cross-section with side-lengths $b = 0.0775$ m. The beam is homogeneous and isotropic elastic with parameters $E = 6.67$ GPa and $G = 2.00$ GPa and mass density $\rho = 200$ kg/m³. The angular velocity is increased over a period $T_s = 15$ s to its final value $\Omega_s = 6$ rad/s according to

$$\Omega_2(t) = \begin{cases} \frac{\omega_s}{T_s} \left[t - \frac{T_s}{2\pi} \sin \frac{2\pi t}{T_s} \right], & 0 \leq t \leq T_s \\ \omega_s, & t > T_s \end{cases} \quad (12)$$

Results illustrating the transverse displacement u_1 and the axial displacement u_3 of the tip as function of time are illustrated in Fig. 2 (a) and (b). The transverse displacement shows the characteristic backward bending during the acceleration phase and corresponds closely to the results obtained in [6]. However, due to the applied small displacement beam formulation the results lack in capturing the apparent axial shortening during the acceleration phase, whereas the small elongation due to the centrifugal force in the final near stationary phase is present. The effect of shortening due to bending may be reproduced by a simple post-processing of the results using the beam column theory from [7] as illustrated by a dashed line in Fig. 2 (b).

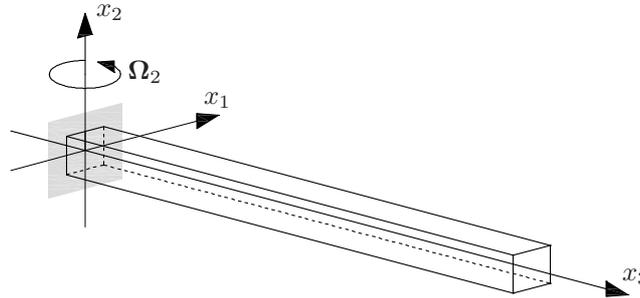


Figure 1: Prismatic beam rotating about a fixed axis.

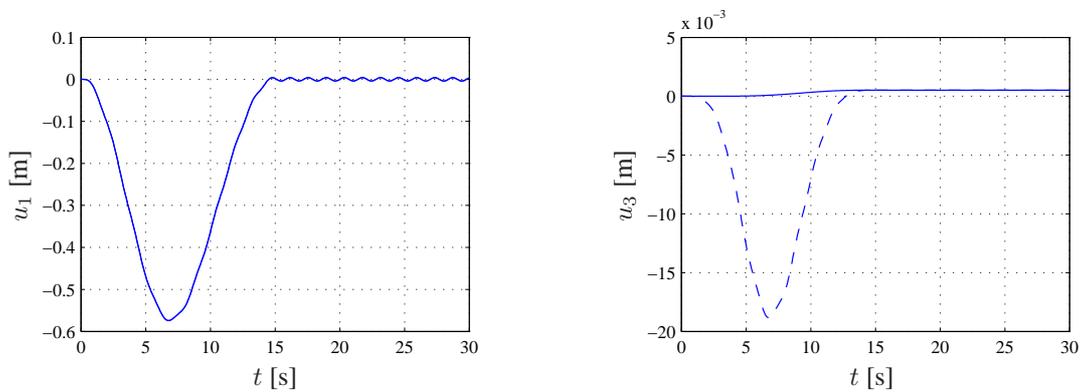


Figure 2: (a) Transverse tip displacement u_1 . (b) Axial tip displacement u_3 . Small displacement theory (—). Beam-column approximation (---).

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