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TUNED MASS ABSORBERS ON DAMPED STRUCTURES

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ABSTRACT

A design procedure is presented for tuned mass absorbers mounted on structures with structural damping. It is demonstrated that by minor modifications of the spectral density integrals very accurate explicit results can be obtained for the variance of the response to wide band random excitation. It is found that the design can be based on the classic frequency tuning, leading to equal damping ratio for the two modes, and an accurate explicit approximation is found for the optimal damping parameter of the absorber and the resulting damping ratio for the response.

1. INTRODUCTION

Tuned mass absorbers constitute an efficient means of introducing damping into structures prone to vibrations, e.g. bridges and high-rise buildings. The original idea is due to Frahm in 1909, who introduced a spring supported mass, tuned to the natural frequency of the oscillation to be reduced. It was demonstrated by Ormondroyd & Den Hartog [1] that the introduction of a damper in parallel with the spring support of the tuned mass leads to improved behavior, e.g. in the form of amplitude reduction over a wider range of frequencies. The standard reference to the classic tuned mass absorber is the textbook of Den Hartog [2] describing optimal frequency tuning for harmonic load, and the procedure of Brock [3] for the optimal damping. A detailed analysis of the frequency response properties of the tuned mass absorber has recently been presented by Krenk [4] who demonstrated that the classic frequency tuning leads to equal damping ratio of the two complex modes resulting from the coupled motion of the structural mass and the damper mass. An optimal damping ratio of the absorber was determined that improves on the classic result of Brock. These results are all based on a frequency response analysis, where a root locus analysis can be used to determine the complex natural frequencies of the modes and thereby the damping ratio, while optimal response characteristics are obtained by consideration of the frequency response diagrams for the response amplitude. The results can be obtained in explicit form only when the original structure is undamped. Results including structural damping have been obtained by Fujino & Abe [5] via a perturbation analysis based on the undamped case.

Many of the vibration problems involving tuned mass dampers involve random loads, e.g. due to wind or earthquakes. In the random load scenario the response is characterized by its variance, determined as a moment of the spectral density. This leads to a different approach to the determination of optimal parameters and also to somewhat different optimal parameter values. The two standard problems are a structural mass excited by a random force, and the combined system of structural and damper mass excited by motion of the support of the structure, Fig. 1. Crandall & Mark [6] obtained explicit results for the variance of the response in the case of support acceleration represented by a stationary white noise process, while Jacquot & Hoppe [7] treated the corresponding force excitation problem. In the design of tuned mass dampers the mass, the stiffness and the applied damping must be selected, and the standard procedure is to select suitable stiffness and applied damping, once the mass ratio has been selected. This problem was studied by Warburton & Ayorinde [8–10] under the assumption that the initial structure is undamped. The approach was to derive optimal values of frequency tuning and applied damping for a given mass ratio that minimize the response variance. It turns out that the optimal frequency tuning and level of applied damping depends on the type of random loading and are not identical with the parameters determined for harmonic load.

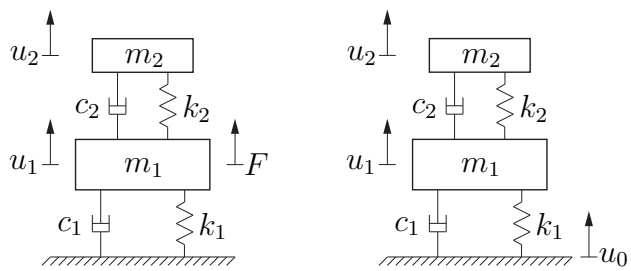

Figure 1. Tuned mass absorber. a) Force excitation, b) Support excitation.

While analytic expressions can be obtained for the response variance also for systems with structural damping [6, 7], it has turned out to be quite difficult to reduce these expressions to explicit design oriented formulae including the structural damping. Surveys of various series expansions have been given by Soong & Dargush [11] and Asami et al. [12]. However, in spite of the large number of terms, these series capture the effect of the structural damping in a fairly indirect way, and it is desirable to find a different format for the combined influence of structural and applied damping. Here it is demonstrated that, while the exact optimal frequency tuning under random load is different from the classic tuning for harmonic load, the influence of this difference on the resulting response variance will in most cases be negligible, if the applied damping is optimized. It is furthermore shown that the damping of the mass absorber can be optimized in terms of the mass ratio alone, without influence from the structural damping, leading to a compact expression for the combined effect of structural and optimal applied damping. A more detailed account can be found in Krenk & Høgsberg [13].

2. PROBLEM FORMULATION

The problem under consideration is illustrated in Fig. 1. The system consists of a structural mass m_1, supported by a spring with stiffness k_1 and a viscous damper with parameter c_1. The motion of the structural mass relative to the ground is denoted $u_1(t)$. The structural mass supports a damper mass m_2 via a spring with stiffness k_2 and a viscous damper with parameter

c_2. The motion of the damper relative to the ground is denoted $u_2(t)$. Two situations will be investigated: a) motion due to a force $F(t)$ acting on the structural mass, and b) motion due to acceleration $\ddot{u}_0(t)$ of the supporting ground.

The motion of the two-degree-of-freedom system is described by a frequency analysis with angular frequency ω and displacement amplitude vector $\mathbf{u} = [u_1, u_2]^T$,

$$\big(\mathbf{K} + i\omega\mathbf{C} - \omega^2\mathbf{M}\big)\,\mathbf{u} = \mathbf{F} \tag{1}$$

where $\mathbf{F} = [F_1, F_2]^T$ is the force amplitude vector, and the mass, damping and stiffness matrices are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \;,\; \mathbf{C} = \begin{bmatrix} c_1{+}c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \;,\; \mathbf{K} = \begin{bmatrix} k_1{+}k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{2}$$

The stiffness parameters are expressed in terms of representative frequencies,

$$\omega_1 = \sqrt{k_1/m_1} \quad , \quad \omega_2 = \sqrt{k_2/m_2} \tag{3}$$

and the damping parameters are expressed in terms of the damping ratios,

$$2\zeta_1 = \frac{c_1}{\sqrt{k_1\, m_1}} \quad , \quad 2\zeta_2 = \frac{c_2}{\sqrt{k_2\, m_2}} \tag{4}$$

The relative mass and time scale of the secondary mass are described by the mass ratio μ and the frequency tuning parameter α,

$$\mu = \frac{m_2}{m_1} \qquad , \qquad \alpha = \frac{\omega_2}{\omega_1} \tag{5}$$

In the typical tuned mass design problem the final damping is controlled by μ, and optimal properties are obtained by proper selection of the frequency tuning parameter α and the damping ratio ζ_2.

Analytic results are obtained for the idealized case of white noise, representing wide-band excitation. The quality of the damper system is defined via its ability to limit the variance of the response of the primary mass, $\sigma_1^2 = \mathrm{Var}[u_1]$. The analysis is therefore based on the frequency transfer function for the component u_1 alone. It is convenient to introduce the frequency ratio $r = \omega/\omega_1$ and to introduce the normalized force $\mathbf{f} = \mathbf{F}/k_1$, whereby (1) takes the form

$$\begin{bmatrix} 1{+}\mu\alpha^2{-}r^2{+}\,2i(\zeta_1{+}\mu\alpha\zeta_2)r & -\mu\alpha^2{-}2i\mu\alpha\zeta_2 r \\ -\mu\alpha^2{-}2i\mu\alpha\zeta_2 r & \mu\alpha^2{-}\mu r^2{+}2i\mu\alpha\zeta_2 r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{6}$$

Only special load processes will be treated here, and it is therefore convenient to discuss the two cases separately.

3. FORCED MOTION

In the case of forced motion there is only one load vector component $F(t)$, acting on the structural mass. The corresponding normalized load vector is,

$$\mathbf{f} = \begin{bmatrix} f \\ 0 \end{bmatrix} = \begin{bmatrix} F/k_1 \\ 0 \end{bmatrix} \tag{7}$$

where the component $f = F/k_1$ corresponds to the quasi-static displacement of the structural mass. The response u_1 of the structural mass follows from (6) as

$$u_1 \;=\; H_f(\omega)\, f \tag{8}$$

The frequency response function $H_f(\omega)$ of the forced motion is a rational function of the form

$$H_f(\omega) \;=\; \frac{p_f(\omega)}{q(\omega)} \tag{9}$$

with numerator

$$p_f(\omega) \;=\; \alpha^2 - r^2 + 2i\alpha\zeta_2 r \tag{10}$$

and denominator

$$q(\omega) \;=\; [\,1+\mu\alpha^2 - r^2 + 2i(\zeta_1+\mu\alpha\zeta_2)r\,][\,\alpha^2 - r^2 + 2i\alpha\zeta_2 r\,] \;-\; \mu[\,\alpha^2 + 2i\alpha\zeta_2 r\,]^2 \tag{11}$$

Let the force be represented by a white noise process, and let the normalized force process $f(t) = F(t)/k_1$ have the spectral density S_f. The variance of the response σ_1^2 can then be evaluated from the integral,

$$\sigma_1^2 \;=\; S_f \int_{-\infty}^{\infty} |H_f(\omega)|^2\, d\omega \;=\; S_f \int_{-\infty}^{\infty} H_f(\omega)\, H_f(-\omega)\, d\omega \tag{12}$$

The poles of the rational frequency transfer function $H_f(\omega)$ all lie in the upper complex half-plane. This type of integral can be evaluated directly from the coefficients of the polynomials in the numerator and denominator [14],

$$\sigma_1^2 \;=\; \frac{\pi}{2}\, S_f\, \omega_1\, \frac{P_f(\mu,\alpha,\zeta_1,\zeta_2)}{Q(\mu,\alpha,\zeta_1,\zeta_2)} \tag{13}$$

where P_f and Q are polynomials in the coefficients of $p_f(r)$ and $q(r)$, respectively, and thereby in the indicated arguments. After some algebra the result can be written as

$$P_f \;=\; \mu\alpha^2(\alpha\zeta_1+\zeta_2) + [\,1-(1+\mu)\alpha^2]^2\zeta_2 + 4\alpha\zeta_2[\zeta_1+(1+\mu)\alpha\zeta_2](\alpha\zeta_1+\zeta_2) \tag{14}$$

and

$$Q \;=\; \mu\alpha(\alpha\zeta_1+\zeta_2)^2 + [\,1-(1+\mu)\alpha^2]^2\zeta_1\zeta_2 + 4\alpha\zeta_1\zeta_2\,[\,\zeta_1+(1+\mu)\alpha\zeta_2](\alpha\zeta_1+\zeta_2) \tag{15}$$

It is seen that all terms in P_f are linear or cubic in the damping ratios ζ_1 and ζ_2, while all terms in Q are quadratic or quartic. In spite of this property, and the fact that several of the factors occur repeatedly, the general form of the exact result seems to be intractable analytically. In the following the result will be analyzed in two steps. First an analysis of the system without structural damping is used to demonstrate that the frequency tuning corresponding to a bifurcation point in the root locus diagram can be used as a fairly good representative for optimal frequency tuning in the case of random load, and subsequently simple and quite accurate approximations for the damping parameters and properties.

3.1 Undamped primary structure

In the absence of structural damping, $\zeta_1 = 0$, the expression (13) for the response variance simplifies considerably,

$$\sigma_1^2 \;=\; \frac{\pi}{2}\,\frac{S_f\omega_1}{\alpha\zeta_2}\,\Big[\,\alpha^2 + \frac{1}{\mu}\big(\,1-(1+\mu)\alpha^2\big)^2 + \frac{4}{\mu}(1+\mu)(\alpha\zeta_2)^2\,\Big] \tag{16}$$

The minimum value of the variance is conveniently found by considering minimizing this expression with the parameter combinations $\alpha\zeta_2$ and α^2 as independent variables. This leads to the optimal frequency tuning ratio α and optimal damping ratio ζ_2 determined by

$$\alpha^2 \;=\; \frac{1+\frac{1}{2}\mu}{(1+\mu)^2} \qquad , \qquad \zeta_2^2 \;=\; \frac{\mu}{4(1+\mu)}\,\frac{1+\frac{3}{4}\mu}{1+\frac{1}{2}\mu} \tag{17}$$

The minimum response variance is found by substituting these optimum values into the response variance expression (16),

$$\sigma_{1,min}^2 \;=\; 2\pi S_f\omega_1\sqrt{\frac{1+\frac{3}{4}\mu}{\mu(1+\mu)}} \tag{18}$$

These are the classic expressions for optimal frequency and damping, and resulting response variance [6, 8, 10].

Alternatively the frequency tuning can be selected as

$$\alpha \;=\; \frac{1}{1+\mu} \tag{19}$$

corresponding to a bifurcation point in the root locus diagram and equal damping of the two modes of the system, [4]. When using this frequency tuning, the optimal damping ratio follows from minimizing (16) as

$$\zeta_2 \;=\; \frac{\sqrt{\mu}}{2} \tag{20}$$

The corresponding value of the response variance is

$$\sigma_1^2 \;=\; \frac{2\pi}{\sqrt{\mu}}\,S_f\omega_1 \;=\; \frac{\pi}{\zeta_2}\,S_f\omega_1 \tag{21}$$

These expressions are remarkably simple. Furthermore the ratio of the minimum standard deviation $\sigma_{1,min}$ to this value is

$$\frac{\sigma_{1,min}}{\sigma_1} \;=\; \sqrt[4]{\frac{1+\frac{3}{4}\mu}{1+\mu}} \;\simeq\; 1-\tfrac{1}{16}\mu+\cdots \tag{22}$$

In most practical cases the mass ratio is of the order of a few percent, leading to a relative difference in the standard deviation of the response of the order 0.001. In view of this the simple frequency tuning $\alpha = (1+\mu)^{-1}$ is used as basis for the development of an approximate but accurate set of formulae for the general case of random load.

3.2 Damped primary structure

In the case of the classic frequency tuning for harmonic load (19) the rational expression (13) for the response variance simplifies. The polynomial in the numerator now takes the form

$$P_f \;=\; \mu\alpha(\alpha^2\zeta_1 + \zeta_2) + 4\alpha\zeta_2(\zeta_1 + \zeta_2)(\alpha\zeta_1 + \zeta_2) \tag{23}$$

When the mass ratio is small, the frequency tuning parameter is close to 1, and for optimal damping the total damping will be in the order of $\sqrt{\mu}$. In typical applications the structural damping ζ_1 will furthermore be small relative to the applied damping ζ_2. Under these conditions exchange of the factor α^2 in the first parenthesis with α will have only modest effect on the numerical value, while leading to a factored form. The polynomial in the denominator of (13) can be factored by a similar approximation. The classic harmonic frequency tuning (13) gives

$$Q \;=\; \mu\alpha\big[\,\alpha^2\zeta_1^2 + \zeta_2^2 + (1+\alpha)\zeta_1\zeta_2\,\big] + 4\alpha\zeta_1\zeta_2\,(\zeta_1 + \zeta_2)(\alpha\zeta_1 + \zeta_2) \tag{24}$$

Again, replacement of the factor α^2 with α in the first term leads to a factored form. When the approximate factored forms are used in the expression (13) for the response variance, the following simple expression is obtained

$$\sigma_1^2 \;\simeq\; \frac{\pi}{2}\frac{S_f\omega_1}{\zeta_1 + \zeta_2}\,\frac{\mu + 4\zeta_2(\zeta_1 + \zeta_2)}{\mu + 4\zeta_1\zeta_2} \tag{25}$$

This approximation contains the exact result in both the limit of vanishing structural damping, $\zeta_1 = 0$, and in the absence of imposed damping, $\zeta_2 = 0$.

For a general combination of damping its effect can be expressed in terms of an effective damping ratio ζ_{eff}, defined by analogy with the formula for structural damping alone, as

$$\sigma_1^2 \;=\; \frac{\pi}{2}\frac{S_f\omega_1}{\zeta_{eff}} \tag{26}$$

It follows from (25) that the effective damping is given by

$$\zeta_{eff} \;=\; \zeta_1 + \frac{\mu\,\zeta_2}{\mu + 4\zeta_2(\zeta_1 + \zeta_2)} \tag{27}$$

Minimum response variance is obtained for maximum effective damping, and thus the last term in (27) should be maximized. When minimizing its reciprocal, the result can be read off directly as

$$\zeta_{2,opt} \;=\; \frac{\sqrt{\mu}}{2} \qquad , \qquad \zeta_1 \geq 0 \tag{28}$$

This leads to the interesting conclusion that the magnitude of the optimal applied damping depends only on the mass ratio, but is independent of the structural damping. The corresponding effective damping ratio is

$$\zeta_{eff,opt} \;=\; \zeta_1 + \frac{\zeta_2^2}{\zeta_1 + 2\zeta_2} \;=\; \frac{(\zeta_1 + \zeta_2)^2}{\zeta_1 + 2\zeta_2} \tag{29}$$

with the optimized response variance

$$\sigma_{1,opt}^2 \;=\; \frac{\pi}{2}\,\frac{\zeta_1 + 2\zeta_2}{(\zeta_1 + \zeta_2)^2}\,S_f\omega_1 \tag{30}$$

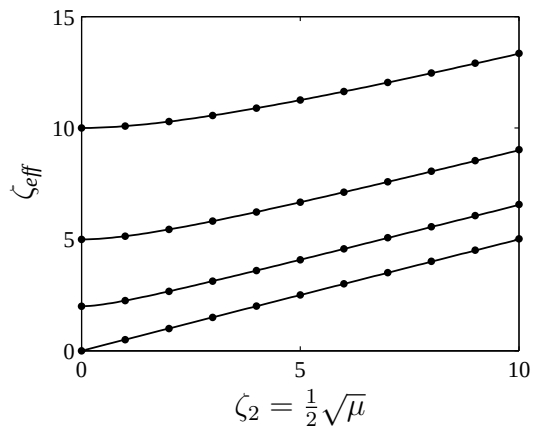

Figure 2: Effective damping ratio ζ_{eff} in % for force excitation. Full lines for explicit approximation and dots for optimal numerical solution for given μ. Structural damping: $\zeta_1 = 0\%$, 2%, 5% and 10%.

It is most convenient to illustrate the combined effect of structural and applied damping in terms of the effective damping ratio. Figure 2 shows the development of the effective damping ratio as a function of the applied damping ζ_2 for different values of the structural damping ζ_1. Note, that for $\zeta_2 = 0$ the effective damping is equal to the structural damping, and thus the structural damping for each curve can be read off by its intersection with the vertical axis. The fully drawn curves give the results from the approximate formula (29) where ζ_2 is defined from the mass ratio by (28). The dots indicate what the effective damping would be, if this mass ratio was given, and frequency tuning α as well as damping ratio ζ_2 were then optimized to find the precise minimum of the response variance σ_1^2. The optimal values were found by a simple numerical search. It is seen that the approximate procedure consisting in use of classic frequency tuning, followed by optimized damping ζ_2 by (28) and the approximate formula (29) gives a response variance that is just about indistinguishable from the exact minimum value.

The curve for zero structural damping is the straight line $\zeta_{eff} = \frac{1}{2}\zeta_2$, also known from the case of harmonic loading [4]. The curves for cases including finite structural damping appear to exhibit asymptotic behavior for increasing applied damping ζ_2 parallel to this line but at a slightly lower level than suggested by the initial value of structural damping alone. An explicit asymptotic formula can be found by writing the formula (29) for the optimal effective damping in the alternative form

$$\zeta_{eff,opt} = \zeta_1 + \tfrac{1}{2}\zeta_2 - \frac{1}{2}\frac{\zeta_1\zeta_2}{\zeta_1 + 2\zeta_2} \tag{31}$$

It follows from this formula that for the typical case of $\zeta_2 \gg \zeta_1$ the last term contributes $-\frac{1}{4}\zeta_1$, leaving the effective damping as

$$\zeta_{eff,opt} \simeq \tfrac{3}{4}\zeta_1 + \tfrac{1}{2}\zeta_2 \quad , \quad \zeta_1 \ll \zeta_2 \tag{32}$$

Thus, in the typical case of relatively small structural damping it contributes with $\frac{3}{4}\zeta_1$, while the applied damping contributes $\frac{1}{2}\zeta_2$ to the combined effective damping.

4. SUPPORT ACCELERATION

When the system is loaded via support acceleration the total motion is $\mathbf{u} + \mathbf{u}_0$, where $\mathbf{u}_0$ represents the support motion. The support motion leads to translations that do not directly activate

elastic and damping forces. Thus, the support motion only contributes to the inertial term, and the equation of motion can be expressed in the form (1) with an equivalent load vector

$$\mathbf{F} \;=\; -\mathbf{M}\,\ddot{\mathbf{u}}_0 \;=\; -\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \ddot{u}_0 \tag{33}$$

It is important to note that the response is calculated for a white noise representation of the support acceleration process $\ddot{u}_0$. The corresponding normalized force is obtained by division with the stiffness k_1 of the primary structure, whereby

$$\mathbf{f} \;=\; -\begin{bmatrix} 1 \\ \mu \end{bmatrix} \frac{\ddot{u}_0}{\omega_1^2} \tag{34}$$

It is convenient to consider the normalized acceleration $\ddot{u}_0/\omega_1^2$ as input in the following to obtain the most direct analogy with the case of force excitation. The response u_1 of the structural mass follows from (6) as

$$u_1 \;=\; H_a(\omega)\,\ddot{u}_0/\omega_1^2 \tag{35}$$

The frequency response function $H_a(\omega)$ of motion due to support acceleration then follows in the form

$$H_a(\omega) \;=\; \frac{p_a(\omega)}{q(\omega)} \tag{36}$$

The numerator is found as

$$p_a(\omega) \;=\; (1{+}\mu)\alpha^2{-}r^2{+}2i(1{+}\mu)\alpha\zeta_2 r \tag{37}$$

while the denominator is the same as in the case of force excitation, already given in (11).

Let the support acceleration be represented by a white noise process, and let the normalized support acceleration $\ddot{u}_0/\omega_1^2$ have the spectral density S_a. The system can be analyzed and reduced in a manner similar to that used for the forced response, [13]. For an undamped primary structure, $\zeta_1 = 0$, the optimal absorber parameters are obtained by minimizing the structural response σ_1^2,

$$\alpha^2 \;=\; \frac{1+\frac{1}{2}\mu}{(1-\mu)^2} \qquad , \qquad \zeta_2^2 \;=\; \frac{\mu}{4(1+\mu)}\,\frac{1-\frac{1}{4}\mu}{1-\frac{1}{2}\mu} \tag{38}$$

When structural damping is included, $\zeta_1 > 0$, a simplified approximate expression for the response variance can be obtained by omission of 'small' terms be obtained in the form

$$\sigma_1^2 \;\simeq\; \frac{\pi}{2}\frac{S_a\omega_1(1+\mu)}{\zeta_1+\zeta_2}\,\frac{\dfrac{\mu}{1+\mu}+4\zeta_2(\zeta_1+\zeta_2)}{\dfrac{\mu}{1+\mu}+4\zeta_1\zeta_2} \tag{39}$$

This expression is similar to (25) for the case of force excitation, when a factor $(1+\mu)$ is included in the spectral density of the excitation, and the mass ratio is represented by $\mu/(1+\mu)$ in the last term. These changes correspond to the fact that in the present case the load acts on the total mass of structure and absorber. This implies that the optimal value of the applied damping follows from (28) by a simple parameter replacement,

$$\zeta_{2,opt} \;=\; \frac{1}{2}\sqrt{\frac{\mu}{1+\mu}} \qquad , \qquad \zeta_1 \geq 0 \tag{40}$$

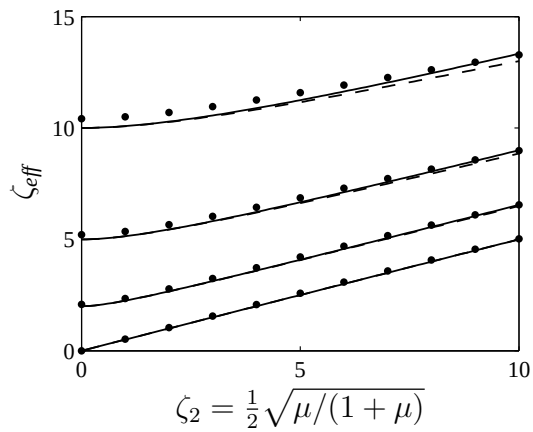

Figure 3: Effective damping ratio ζ_{eff} in % for support acceleration. Full lines for explicit approximation and dots for optimal numerical solution for given μ. Structural damping: $\zeta_1 = 0\%$, 2%, 5% and 10%.

In the case of support acceleration it is convenient to define the effective damping by the relation

$$\sigma_1^2 \;=\; \frac{\pi}{2}(1+\mu)\frac{S_a\omega_1}{\zeta_{eff}} \tag{41}$$

When using the optimal applied damping (40) the corresponding effective damping ratio $\zeta_{eff,opt}$ is given by (29) as for forced response. The approximate results (42) can therefore be illustrated graphically in Fig. 3 in the same way as for the forced response. It is noted that the optimal absorber damping now is expressed in terms of $\mu/(1+\mu)$ instead of μ. The approximate results are slightly less accurate in this case, but for realistic structural damping ratio $\zeta_1 < 0.05$ they are very good over the full range of the absorber damping ratio ζ_2.

5. EXAMPLE

Consider damping of a 10-storey shear frame structure with a tuned mass damper attached to the top floor, Fig. 4. The concentrated mass of each floor is $m = 1$ and the interstorey stiffness k is chosen so that $k/m = 100$, corresponding to the lowest natural angular frequency $\omega_1 = 1.495$. Structural damping is introduced by Rayleigh type damping with mass proportional factor 0.0258 and stiffness proportional factor 0.0039. This provides equal modal damping ratios of 0.0115 for the first two modes, while the damping ratio for mode 10 is 0.0390. The mass ratio of the tuned mass absorber is calculated for mode i by using the modal mass, and the corresponding modal mass of the absorber,

$$m_i \;=\; \boldsymbol{\varphi}_i^T\mathbf{M}\boldsymbol{\varphi}_i \qquad , \qquad m_a \;=\; \boldsymbol{\varphi}_i^T\mathbf{M}_a\boldsymbol{\varphi}_i \tag{42}$$

where $\boldsymbol{\varphi}_i$ is the mode shape vector, $\mathbf{M}$ is the mass matrix of the structure, and $\mathbf{M}_a$ is the mass matrix of the absorber mass located at the corresponding node of the structure, see [16].

Two idealized load cases are considered: wind excitation and ground acceleration. For wind excitation the mass absorber is tuned by the expressions associated with forced response, given in (17) for the optimal design without structural damping and in (19) and (28) for the approximate design with structural damping. For ground excitation the mass absorber is tuned according to the expressions obtained for ground acceleration, i.e. (38) for the optimal design without

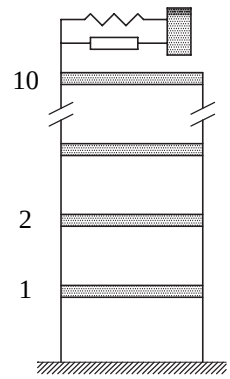

Figure 4. 10-storey shear frame with tuned mass absorber on top floor.

structural damping and (19) and (39) for the approximate design with structural damping. Table 1 summarizes the parameters for the various designs of the tuned mass absorber, and gives the natural angular frequencies and damping ratios for mode 1, obtained by solving the complex eigenvalue problem. Two natural frequencies and damping ratios are associated with mode 1 since the tuned mass damper introduces an additional degree of freedom. It is found that the approximate tuning leads to an almost equal split of the damping ratio into the two modes, as implied by the nature of the tuning principle. The tuning that is optimal in the case without structural damping leads to a larger damping of one mode and thereby less damping of the other mode, which will therefore appear as the critical mode. Although the differences are small the equal damping property of the approximate tuning introduces a desirable robustness.

The efficiency of the tuned mass absorber is also verified by simulations. The wind excitation is approximated by unit Gaussian processes acting independently on each floor, but not on the tuned mass absorber. For the ground acceleration the acceleration process is also generated as a unit Gaussian process. For all simulations the time increment is $\Delta t = 0.05$. The load is constant over each time step. For this type of process the frequency dependent spectral density of the Gaussian process S_f relative to the corresponding white noise level S_0 is given as

Table 1. Tuned mass absorber parameters and mode 1 properties.

			optimal				approximate			
	μ	m_a	ω_a	ζ_a	ω_1	ζ_1	ω_a	ζ_a	ω_1	ζ_1
wind	0.02	0.106	1.473	0.070	1.390	0.039	1.465	0.071	1.387	0.041
					1.580	0.043			1.577	0.042
	0.05	0.264	1.441	0.110	1.326	0.058	1.423	0.112	1.317	0.062
					1.618	0.064			1.608	0.062
	0.1	0.528	1.392	0.153	1.250	0.078	1.359	0.158	1.235	0.086
					1.652	0.089			1.633	0.086
ground	0.02	0.106	1.458	0.070	1.383	0.042	1.465	0.070	1.386	0.040
					1.573	0.040			1.577	0.041
	0.05	0.264	1.406	0.110	1.307	0.064	1.423	0.109	1.316	0.060
					1.601	0.058			1.609	0.061
	0.1	0.528	1.324	0.153	1.214	0.089	1.359	0.151	1.233	0.082
					1.620	0.077			1.636	0.082

$S_f/S_0 = [\sin(\frac{1}{2}\omega\Delta t)/(\frac{1}{2}\omega\Delta t)]^2$. Note, that this is a better white-noise representation than the linear interpolation introduced in [15], leading to a spectral density with the power 4 instead of the present 2. For mode 1 the Gaussian process is practically white with $S_f(\omega_1)/S_0 = 0.9995$, whereas for mode 10 the spectral density is slightly reduced, $S_f(\omega_1)/S_0 = 0.9211$. The present time increment $\Delta t = 0.05$ leads to approximately 84 time steps per period of mode 1 and 6 time steps per period of mode 10. Each simulation record contains 10^6 time increments, which corresponds to more than 11000 periods of mode 1.

The response magnitude of the structure is assessed by the accumulated variance,

$$\sigma^2 = \sum_{j=1}^{n} \sigma_j^2 \tag{43}$$

where σ_j^2 is the variance of the jth floor. The accumulated variance is shown in Fig. 5 for wind excitation (left) and ground acceleration (right), where σ_0^2 is the accumulated variance for the structure without tuned mass absorber. In Fig. 5 black bars represent the tuning that is optimal without structural damping, while white bars represent the approximate tuning. It is seen that the efficiency increases with the mass ratio, and that the performance of the two tuning procedures are practically identical.

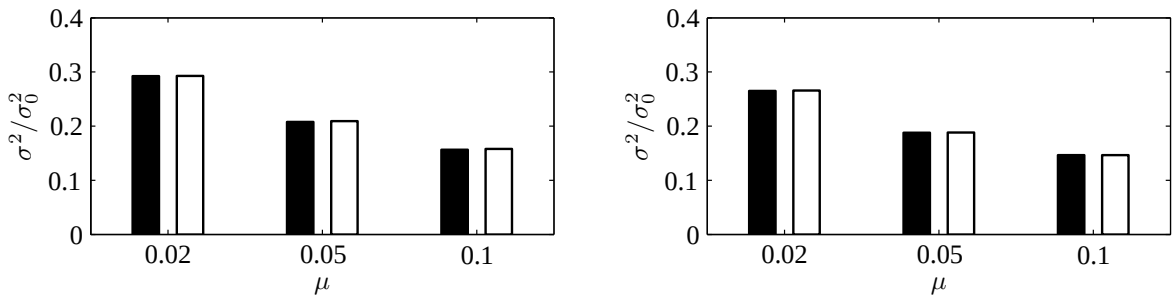

Figure 5: Relative accumulated variance σ^2/σ_0^2. Left: wind excitation and right: ground excitation. Black bars: optimal tuning for $\zeta_1 = 0$; white bars: general approximate tuning.

Figure 6 shows the variance σ_a^2 of the relative absorber mass displacement divided by the variance of the top floor displacement σ_{10}^2. It is seen that the absorber mass response is significantly larger than the structural response. The difference decreases with increasing mass ratio. Again the difference between the two design procedures is negligible.

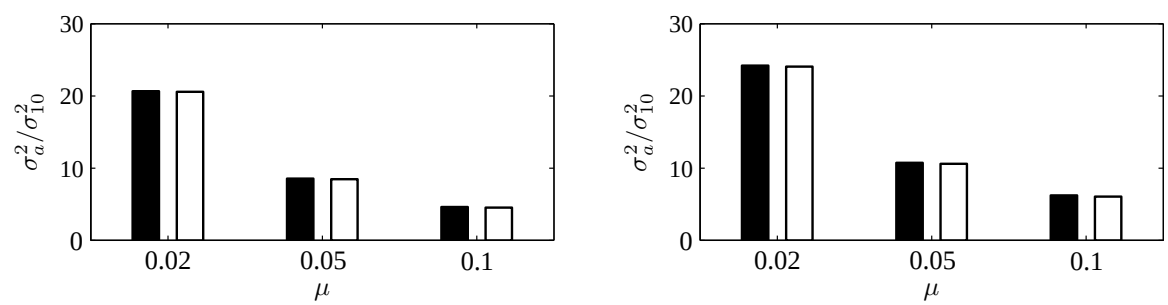

Figure 6: Variance of absorber mass response σ_a^2/σ_{10}^2. Left: wind excitation and right: ground excitation. Black bars: optimal tuning for $\zeta_1 = 0$; white bars: general approximate tuning.

6. CONCLUSIONS

It has been demonstrated that the frequency tuning of tuned mass absorbers on structures under wide-band random load can be selected according to an 'equal modal damping principle valid for harmonic excitation. The damping constant of the absorber is subsequently selected to minimize the resulting modal variance for the selected frequency tuning. This procedure leads to response characteristics practically indistinguishable from the exact optimum, and furthermore leads to the simple explicit formula (29) for the combined effective damping of the structural response modes in the presence of the absorber.

REFERENCES

[1] J. Ormondroyd and J.P. Den Hartog, The theory of the dynamic vibration absorber. *Transactions of ASME*, **50**(APM-50-7), 9–22, 1928.

[2] J.P. Den Hartog, *Mechanical Vibrations* (4th edn). McGraw-Hill, New York, 1956. (Reprinted by Dover, New York, 1985)

[3] J.E. Brock, A note on the damped vibration absorber. *Journal of Applied Mechanics*, **13**, A284, 1946.

[4] S. Krenk, Frequency analysis of the tuned mass damper. *Journal of Applied Mechanics*, **72**, 936–942, 2005.

[5] Y. Fujino and M Abe, Design formulas for tuned mass dampers based on a perturbation technique. *Earthquake Engineering and Structural Dynamics*, **22**, 833–854, 1993.

[6] S.H. Crandall and W.D. Mark, *Random Vibration.* Academic Press, New York, 1967.

[7] R.G. Jacquot and D.L. Hoppe, Optimal random vibration absorbers. *Journal of the Engineering Mechanics Division ASCE*, **99**E3, 936–942, 1973.

[8] E.O. Ayorinde and G.B. Warburton, Minimizing structural vibrations with absorbers. *Earthquake Engineering and Structural Dynamics*, **8**, 219–236, 1980.

[9] G.B. Warburton, Optimum absorber parameters for minimizing vibration response. *Earthquake Engineering and Structural Dynamics*, **9**, 251–262, 1981.

[10] G.B. Warburton, Optimum absorber parameters for various combinations of response and excitation parameters. *Earthquake Engineering and Structural Dynamics*, **10**, 381–401, 1982.

[11] T.T. Soong and G.F. Dargush, *Passive Energy Dissipation Systems in Structural Engineering.* Wiley, Chichester, UK, 1997.

[12] T. Asami, O. Nishihara and A.M. Baz, Analytical solution to H_∞ and H_2 optimization of dynamic vibration absorbers attached to damped linear systems. *Journal of Vibration and Acoustics*, **124**, 284–295, 2002.

[13] S. Krenk and J. Høgsberg, Calibration of tuned mass dampers on damped structures ander random load. *Probabilistic Engineering Mechanics*, 2008. DOI: 10.1016/j.probengmech.2007.04.004.

[14] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products.* Academic Press, New York, 1980.

[15] R.W. Clough and J. Penzien, *Dynamics of Structures.* McGraw-Hill, New York, 1975.

[16] S. Krenk, A. Brønden and A. Kristensen, Placement and tuning of resonance dampers on footbridges, *FOOTBRIDGE 2005, 2nd International Conference*, Venice, Italy, December 6-8, 2005.