the first kind at an amplitude of  $\phi = \pi/2$ , thus

$$\mathbf{K} = \mathbf{K}(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1 - t^2)(1 - k^2 t^2)}},\tag{42}$$

where the modulus may be omitted, if no misunderstanding is possible. With the complementary modulus  $k' \equiv \sqrt{1-k^2}$  the complementary complete elliptic integral of the first kind becomes

$$K' = K'(k) = K(k') = K(\sqrt{1-k^2}).$$
 (43)

Legendres elliptic integral of the second kind  $E(\phi, k)$  is defined from the definite integral

$$E(\phi,k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta} \, \mathrm{d}\theta = \int_0^{\sin \phi} \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}} \, \mathrm{d}t \tag{44}$$

$$E_1(z,k) = \int_0^z \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} \,\mathrm{d}t, \text{ with } z = \sin\phi, \tag{45}$$

again the complete elliptic integral E(k) of the second kind is obtained at an amplitude of  $\phi = \pi/2$ 

$$\mathbf{E} = \mathbf{E}(k) = E\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, \mathrm{d}\theta = \int_0^1 \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}} \, \mathrm{d}t.$$
(46)

Jacobi's elliptic integral of the third kind  $\Pi_{J}(z, \alpha, k)$  is the definite integral

$$\Pi_{\mathbf{J}}(z,\alpha,k) = k^2 \operatorname{sn}(\alpha,k) \operatorname{cn}(\alpha,k) \operatorname{dn}(\alpha,k) \int_0^z \frac{\operatorname{sn}^2(u,k)}{1 - k^2 \operatorname{sn}^2(\alpha,k) \operatorname{sn}^2(u,k)} \mathrm{d}u$$
(47)

which differs from Legendres elliptic integral of the third kind  $\Pi(z, \alpha, k)$ , since that integral in Jacobi's form is defined as

$$\Pi(z, \alpha, k) = \int_0^z \frac{\mathrm{d}u}{1 - k^2 \mathrm{sn}^2(\alpha, k) \mathrm{sn}^2(u, k)}$$
  
= 
$$\int_0^z \frac{\mathrm{d}t}{(1 - k^2 \mathrm{sn}^2(\alpha, k) t^2) \sqrt{1 - t^2} \sqrt{1 - k^2 t^2}}.$$
 (48)

It follows that the two elliptic integrals of the third kind are related

$$\Pi(z, \alpha, k) = z + \frac{\operatorname{sn}(\alpha, k)}{\operatorname{cn}(\alpha, k) \operatorname{dn}(\alpha, k)} \Pi_{\mathrm{J}}(z, \alpha, k) \,. \tag{49}$$

Jacobi's zeta function Z(u,k) = Z(u) is related to the incomplete elliptic integrals of the first and second kind by

$$Z(u,k) = Z(u) = E_1(u,k) - F_1(u,k) \frac{E(k)}{K(k)}.$$
(50)

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