

The ensemble variance of pure-tone measurements in reverberation rooms^{a)}

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Reverberation rooms are often used for measuring the sound power emitted by sources of sound. At medium and high frequencies, where the modal overlap is high, a fairly simple model based on sums of waves from random directions having random phase relations gives good predictions of the ensemble statistics of measurements in such rooms. Below the Schroeder frequency, the relative variance is much larger, particularly if the source emits a pure-tone. The established theory for this frequency range is based on ensemble statistics of modal sums and requires knowledge of mode shapes and the distribution of modal frequencies. This paper extends the far simpler random wave theory to low frequencies. The two theories are compared, and their predictions are found to compare well with experimental and numerical results.

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I. INTRODUCTION

Statistical models are well established in room acoustics. For example, more than half a century ago Schroeder¹ developed a stochastic model that predicts how the sound pressure at a given position in a reverberant room varies with the frequency. A closely related stochastic theory based on sums of coherent waves arriving from random directions and having random phases gives reliable predictions of the spatial fluctuations of the sound pressure in a reverberant room driven with a pure-tone above the Schroeder frequency.² An alternative theory for this and other phenomena is also stochastic in nature but based on sums of modes with random distributions of the modal frequencies.^{3,4} The modal theory, which is more complicated than the random wave theory and requires more information about the room, is generally regarded as valid also below the Schroeder frequency.⁴ However, there is surprisingly little experimental evidence of its validity in this frequency range.

This paper attempts to extend the simpler theoretical approach to low frequencies. The two theories are compared with experimental and numerical results.

II. OUTLINE OF EXISTING MODELS

The random wave theory is essentially due to Schroeder,¹ Andres,⁵ Waterhouse,² and Lubman.⁶ The alternative modal theory is essentially due to Lyon³ and Davy,^{4,7} but later modified by Weaver.⁸

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A. The random wave theory

Above the Schroeder frequency, a harmonic sound field in a reverberation room can be modeled as a sum of waves

$$p(\mathbf{r}) = \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \sum_{n=1}^N A_n e^{j(\omega t + \mathbf{k}_n \cdot \mathbf{r})}, \quad (1)$$

where $p(\mathbf{r})$ is the sound pressure at position \mathbf{r} , A_n is a complex random amplitude the phase angle of which is uniformly distributed between 0 and 2π , and \mathbf{k}_n is a random wave number vector with a uniform distribution over all angles of incidence. Note that no information about the particulars of the room has entered into this stochastic pure-tone diffuse field theory at this stage. It is easy to show that the corresponding expression for the mean square pressure is a sum of two independent squared Gaussian variables (random sums) with zero mean.² A sum of two squared Gaussian variables with zero mean has a chi-square distribution with two degrees of freedom,^{2,9} from which it follows that the relative ensemble variance is 1,

$$\varepsilon^2 \{p_{\text{rms}}^2\} = 1. \quad (2)$$

In this expression, p_{rms}^2 is the time average of the squared sound pressure at a given position in a given room, and ε^2 is the relative ensemble variance (i.e., the ensemble variance normalized by the squared ensemble average). Above the Schroeder frequency, one can expect the same statistics with respect to room, position, and frequency, and a relative spatial variance of unity has been validated experimentally many times, e.g., in Refs. 2 and 10.

About 30 years ago, Jacobsen¹¹ and Pierce¹² independently used Eq. (1) to calculate the ensemble variance of the sound power output of a monopole that emits a pure-tone in

a reverberation room. They showed that the reverberant part of the sound field expressed by Eq. (1) because of the resulting random contribution to the radiation impedance gives rise to a relative variance of the sound power output of

$$\varepsilon^2\{P_{af}\} = \frac{1}{M_s}, \quad (3)$$

where M_s is the statistical modal overlap of the room (the product of the modal density and the statistical bandwidth of the modes). The modal overlap, which is proportional to the square of the frequency and the total absorption area in the room, is large above the Schroeder frequency, and therefore, the sound power emitted by a monopole essentially equals its free field sound power in this frequency range.

B. The modal theory

The original version of the alternative theory, due to Lyon,³ is based on the analytical Green's function in a rectangular room, which is a modal sum.¹² Lyon assumed that the modal frequencies have a Poisson distribution (i.e., are distributed independently of each other) and predicted the ensemble variance of the sound power output of a monopole and of the mean square pressure. Some years later, Davy⁴ extended Lyon's theory by deriving a more general expression of the power transmission function averaged over multiple source and receiver positions, assuming a "nearest neighbor" distribution of the modal frequencies. The assumption of the modal frequencies having a nearest neighbor distribution rather than a Poisson distribution came from evidence of a "modal repulsion" effect already discussed by Lyon.³ In 1989, Weaver⁸ modified Davy's expression so as to take account of a modal frequency spacing described by the Gaussian orthogonal ensemble theory, which is now generally accepted.¹⁵ Finally, in 1990, Davy¹⁴ described and discussed the modified theory. The resulting expression for the relative ensemble variance of the sound power of a monopole became

$$\varepsilon^2\{P_{af}\} = \frac{K-1}{M_s}, \quad (4)$$

where the spatial factor K is the normalized fourth moment (the kurtosis) of the modal functions ψ as follows:

$$K = \frac{E\{\psi^4\}}{E^2\{\psi^2\}}. \quad (5)$$

In a rectangular room, the mode shape ψ is a product of cosines. This gives a value of $(3/2)^3$ for oblique modes. With this value of K , Eq. (4) is similar to but somewhat larger than the prediction given by Eq. (3) ($K-1 \approx 2.38$).

The corresponding expression for the relative ensemble variance of the mean square pressure became^{8,14}

$$\varepsilon^2\{p_{rms}^2\} = 1 + \frac{K^2-3}{M_s}, \quad (6)$$

which asymptotically approaches Eq. (2) for high values of the modal overlap. [With $K=(3/2)^3$ Eq. (6) becomes $\varepsilon^2\{p_{rms}^2\} \approx 1+8.4/M_s$.] According to Davy¹⁵ the right-hand

side of Eq. (6) can be expected to be modified at very low modal overlap

$$\varepsilon^2\{p_{rms}^2\} = 1 + \frac{K^2}{M_s}. \quad (7)$$

Although there is little doubt that there is an increase in the relative variance of the mean square pressure in a frequency region below the Schroeder frequency compared with the asymptotic value of unity at high modal overlap,¹⁶ there is, for some reason, surprisingly little, if any, experimental evidence in direct support of Eqs. (6) and (7).

It should be mentioned that other authors have suggested lower values of K . Weaver⁸ and Lobkis *et al.*¹⁷ are in favor of a value of 3, which corresponds to normally distributed modal amplitudes; experimental and numerical results have led Langley and Brown¹⁸ to suggest a value of 2.7. See discussions in Refs. 8, 13, and 17–19.

C. Extension of the random wave theory

Recently, Jacobsen and Rodríguez Molares²⁰ modified Eq. (3) by taking into account the local increase in the reverberant part of the sound field at the source position due to "weak Anderson localization," as predicted by Weaver and Burkhardt.²¹ Equation (3) now became

$$\varepsilon^2\{P_{af}\} = \frac{F(M_s)}{M_s}, \quad (8)$$

where the "concentration factor" $F(M_s)$ is a function that goes smoothly from 3 to 2 as the modal overlap is increased.¹³ The modified expression was validated experimentally in various reverberation rooms as well as by numerical calculations. As pointed out in Ref. 20, it is interesting and somewhat surprising that the modal overlap enters into a theory that does not make use of the concept of modes. Since Eqs. (8) and (4) are very similar [with $F=2$ and $K=(3/2)^3$] these results also validated the modal prediction. With $F=2$ and $K=3$ the two expressions are identical.

The finite ensemble variance of the sound power emitted by the source at low and medium modal overlaps means that the ensemble average of $|A_n|^2$ varies from outcome to outcome of the stochastic process. Such variations are not taken into account by Eq. (2). Thus one might expect additional variations in the mean square pressure corresponding to the variations in the sound power below the Schroeder frequency. This can be modeled by multiplying the original exponentially distributed mean square pressure by another random variable that represents the relative variations in the emitted sound power. The latter is a normally distributed independent random variable with an average of 1 and a variance given by Eq. (8). It follows that Eq. (2) becomes

$$\begin{aligned} \varepsilon^2\{p_{rms}^2\} &= \varepsilon^2\{x(1+y)\} = \frac{E\{x^2(1+y)^2\}}{E^2\{x(1+y)\}} - 1 \\ &= \frac{E\{x^2\}E\{(1+y)^2\}}{E^2\{x\}E^2\{(1+y)\}} - 1 = \frac{2E\{(1+y)^2\}}{E^2\{(1+y)\}} - 1 \\ &= 2(1+E\{y^2\}) - 1 = 1 + \frac{2F(M_s)}{M_s}, \end{aligned} \quad (9)$$

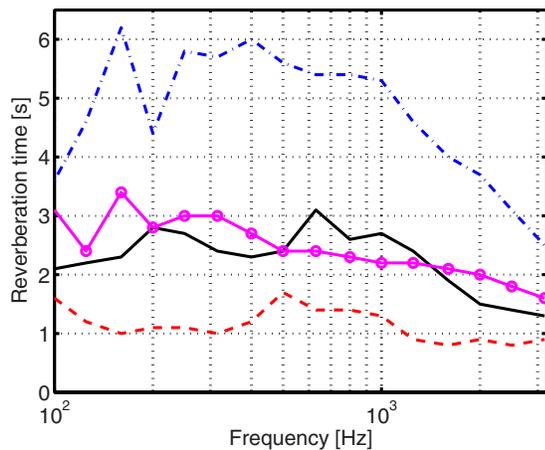


FIG. 1. (Color online) Reverberation time of the test rooms. Solid line: small lightly damped room; dashed line: small damped room; dashed-dotted line: large reverberation room; and line with circle markers: large damped reverberation room.

where x is the original mean square pressure and y is an independent, normally distributed variable with zero mean that represents the relative fluctuations of the emitted sound power. Note that Eq. (9) is similar to Weaver’s equation (6) although the “correction” to the asymptotic high frequency value of unity is somewhat smaller. If $K=3$ is used in Weaver’s expression rather than the value for oblique modes in a rectangular room it becomes $\varepsilon^2\{p_{\text{rms}}^2\} \approx 1 + 6/M_s$, which is quite similar to Eq. (9).

III. EXPERIMENTAL RESULTS

To test the validity of Eqs. (6) and (9) some experiments have been carried out in various rooms at the Technical University of Denmark: a small room (40 m^3) with almost bare walls, the same room with absorption added, a large reverberation room (245 m^3) with very little damping, and the same room with added absorption. Both rooms are essentially rectangular, but there are large (fixed) diffusers in the reverberation room, and the absorption was in all cases irregularly distributed. Figure 1 shows the reverberation time of the rooms, measured in one-third octave bands using the interrupted noise method and a Brüel & Kjær (B&K) “PULSE” analyzer. The Schroeder frequency of the small bare room is about 500 Hz; with added absorption, it is reduced to about 330 Hz. In the large reverberation room, the Schroeder frequency is about 310 Hz; and with added absorption, it is reduced to 200 Hz.

The monopole generating the sound field was a B&K OmniSource fitted with a “volume velocity adapter” with two matched quarter-inch microphones,²² and the sound pressure was measured using the pressure microphone of a “ultimate sound probe” (USP), a three-dimensional pressure-velocity probe produced by Microflown (Zevenaar, The Netherlands). The frequency response between the volume velocity of the source and the sound pressure in the room was measured with the same B&K analyzer but in the fast Fourier transform (FFT) mode, using pseudorandom noise (6400 spectral lines) in the frequency range of up to 3.2 kHz. A similar technique was used recently to validate Eqs. (4)

and (8).²⁰ In order to approach the full variation associated with ensemble statistics both the source and receiver positions were varied, and in the postprocessing of the results (obtained at 25 pairs of positions), additional variations over 8 Hz frequency bands (16 neighboring frequencies) were also taken into account.

Figure 2 compares the results of the measurements of the mean square pressure with predictions calculated using Eq. (6) using a weighted average of K for oblique, tangential, and axial modes, as suggested by Davy,¹⁴ a similar prediction using $K=3$, as suggested by Weaver,⁸ and a prediction calculated using Eq. (9) and $F=2$. As can be seen, the two predictions calculated from Eq. (6) are practically identical, whereas Eq. (9) gives slightly lower values. The measured relative space-frequency standard deviation of the mean square pressure fluctuates somewhat with the frequency, but follows the predicted tendency fairly closely. Above 1 kHz it approaches unity as expected. In general, the experimental data seem to agree equally well with Eqs. (6) and (9).

IV. NUMERICAL RESULTS

It is not practical to determine the full ensemble standard deviation experimentally, but it can be done with a numerical model. In this case, the matter has been examined with the finite element method (FEM), using FEM models of 25 different rooms. The rooms were rectangular and they had a uniform locally reacting wall impedance. The FEM was constructed using the commercial software packet ACTRAN. The dimensions of the rooms were chosen as uniform random variables varying between 2 and 6 m. The source position was placed at random but at least 0.4 m away from any wall. The calculations were carried out from 200 to 300 Hz with a frequency step of 2 Hz. The element size was chosen so as to provide a low numerical pollution in the examined frequency range. The mean square pressure was calculated at 50 000 randomly chosen nodal points of the mesh. Nodes closer than 0.4 m away from the walls or closer than 1 m from the source were not used. In order to determine the relative ensemble standard deviation as a function of the modal overlap, the data were sorted into appropriate modal overlap intervals. A similar technique was used recently in Ref. 20.

In Fig. 3 the results of the FEM calculations are compared with predictions determined using Eq. (9) with $F=2$ and $F=3$. [The latter case is identical to Eq. (6) with $K=3$.] A prediction calculated using Eq. (7) and $K=(3/2)^3$ is also shown. On the whole, the FEM results take values corresponding to Eq. (9) with F between 2 and 3, or Eq. (6) with K between 2 and 3. On the other hand, Eq. (7) overestimates significantly, probably because the modal overlap is too high.

Figure 4 shows the results of similar calculations with the boundary element model (BEM) using the Open Source software “OPENBEM” (University of Southern Denmark, Odense, Denmark) in an ensemble of rectangular two-dimensional “rooms” with uniform wall impedance in the frequency range between 200 and 400 Hz. The same parameters were used as in the FEM calculations, but since there were no nodes in the room, in this case, the mean square pressure was calculated at 1000 random positions at least 0.4

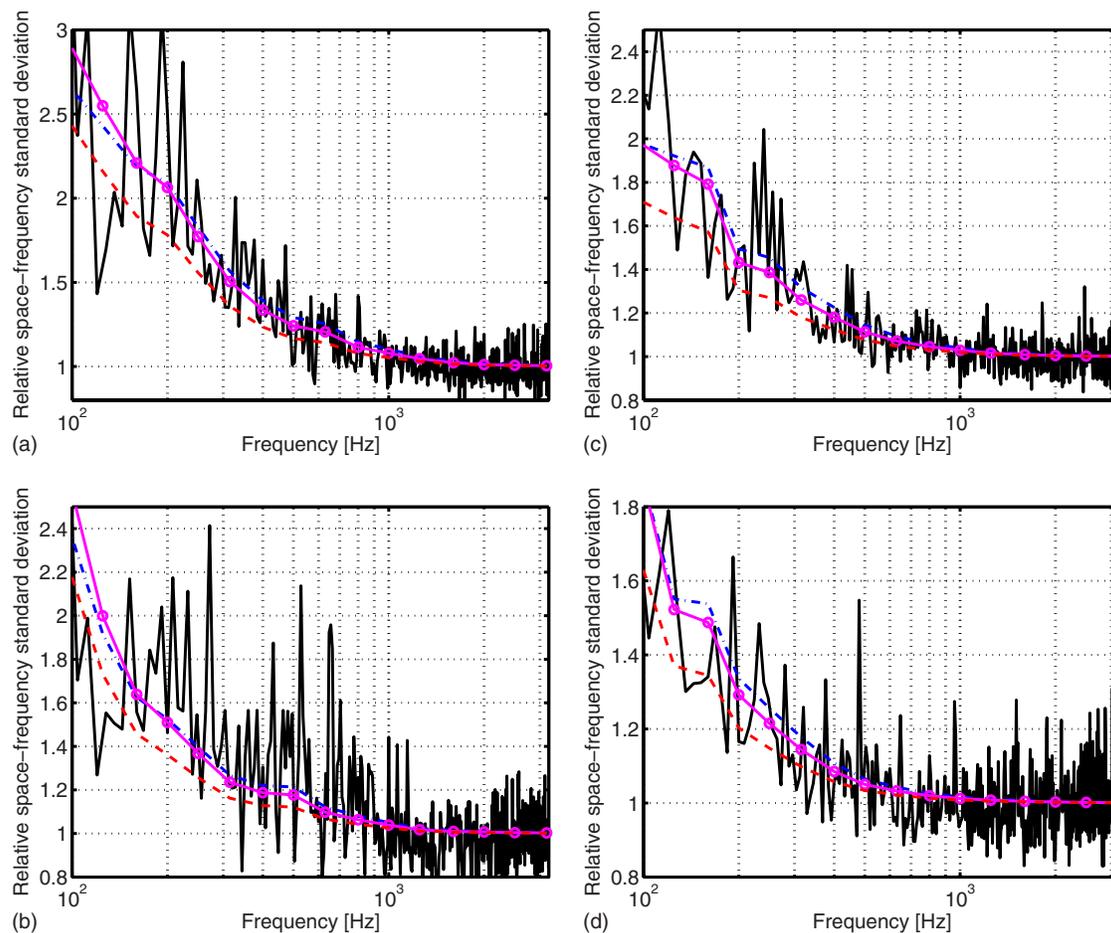


FIG. 2. (Color online) Relative space-frequency standard deviation of mean square pressure in (a) small lightly damped room, (b) small damped room, (c) large reverberation room, and (d) large damped reverberation room. Solid line: measured standard deviation; dashed line: prediction based on Eq. (9) with $F=2$; dashed-dotted line: prediction based on Eq. (6) with a weighted average for K ; and line with circle markers: prediction based on Eq. (6) with $K=3$.

m from the walls and 1 m from the source. The BEM results are compared with predictions determined using Eq. (9) with $F=2$ and 3, and with Eq. (7) and $K=(3/2)^2$. The numerical results are in fairly good agreement with the predictions, particularly those based on Eq. (9) with $F=2$.

V. DISCUSSION

It should be mentioned that the extended random wave theory relies on a result from the modal theory: the concentration factor F . Nevertheless, it is surprising that such a simple theory gives results that, from a practical point of

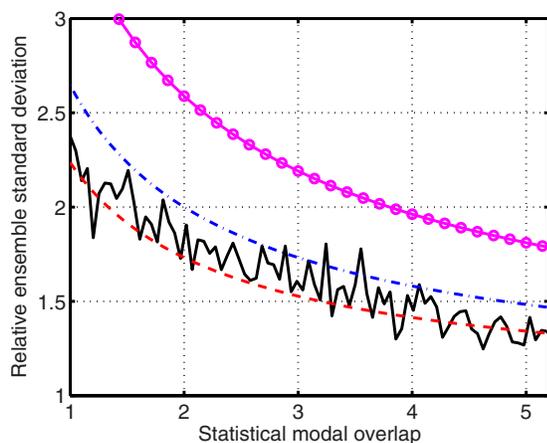


FIG. 3. (Color online) Relative ensemble standard deviation of mean square pressure in three-dimensional rooms. Solid line: FEM calculation; dashed line: prediction based on Eq. (8) with $F=2$; dashed-dotted line: prediction based on Eq. (9) with $F=3$ (or Eq. (6) with $K=3$); and line with circle markers: prediction based on Eq. (7) with $K=(3/2)^3$.

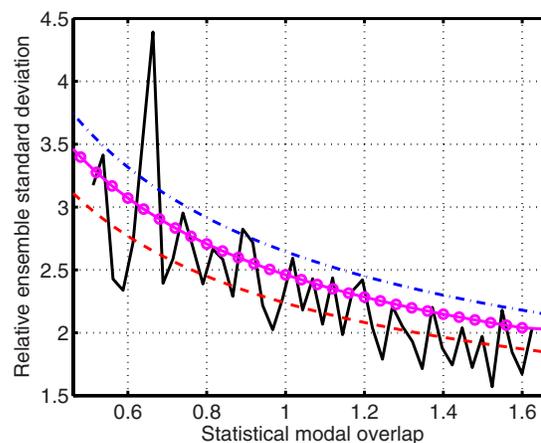


FIG. 4. (Color online) Relative ensemble standard deviation of mean square pressure in two-dimensional “rooms.” Solid line: BEM calculation; dashed line: prediction based on Eq. (9) with $F=2$; dashed-dotted line: prediction based on Eq. (9) with $F=3$ (or Eq. (6) with $K=3$); and line with circle markers: prediction based on Eq. (7) with $K=(3/2)^2$.

view, are simply identical to the results of the far more complicated modal theory. On the other hand, in view of the fact that modes can be decomposed into waves, it is perhaps not *that* surprising that the two approaches lead to comparable ensemble statistics.

Equation (8) is arguably a first order approximation that might be improved by taking into account the increased variance of the reverberant part of the sound pressure predicted by Eq. (9). However, the experimental and numerical results presented in Ref. 20 confirm Eq. (8) and do not support any “higher order correction.”

VI. CONCLUSION

Experimental and numerical results confirm that there is a substantial increase in the relative ensemble variance of the mean square pressure in a reverberation room driven by a monopole that emits a pure-tone below the Schroeder frequency. Above this frequency, the relative variance approaches unity; below this frequency, there is an increase in the variance that is inversely proportional to the modal overlap, that is, proportional to the ratio of the reverberation time to the room volume and inversely proportional to the square of the frequency.

Waterhouse’s simple random wave theory has been extended to the frequency range below the Schroeder frequency and has shown to give predictions of the relative ensemble variance of the mean square pressure in good agreement with the more complicated statistical modal theory due to Lyon, Davy, and Weaver, and these predictions are confirmed by the experimental and numerical results.

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