Predicting Turbulent Flow and Heat Transfer in 3D Ducts Using an EASM

M. Rokni\textsuperscript{1} and T. B. Gatski\textsuperscript{2}

\textsuperscript{1}Division of Heat Transfer, Lund Institute of Technology, 221 00 Lund, Sweden
\textsuperscript{2}Computational Modeling & Simulation Branch, NASA Langley Research Center, Hampton, Virginia 23681, USA

Abstract — The performance of an explicit algebraic stress model is assessed in predicting the turbulent flow and forced heat transfer in straight ducts, with square, rectangular, trapezoidal and triangular cross-sections, under fully developed conditions over a range of Reynolds numbers. Iso-thermal conditions are imposed on the duct walls and the turbulent heat fluxes are modeled by gradient-diffusion type models. At high Reynolds numbers ($>10^5$), wall functions are used for the velocity and temperature fields; while at low Reynolds numbers damping functions are introduced into the models. Hydraulic parameters such as friction factor and Nusselt number are well predicted even when damping functions are used, and the present formulation imposes minimal demand on the number of grid points without any convergence or stability problems.

1. Introduction

The predictive performance of turbulence models in calculating the velocity and temperature fields of turbulent duct flows has become increasingly important during the last few years. Such flows occur frequently in many industrial applications, such as compact heat exchangers, gas turbine cooling systems, cooling channels in combustion chambers, nuclear reactors, etc. The cross-section of these ducts can be both orthogonal (square or rectangular) and non-orthogonal (such as trapezoidal) in which the generated flow is extremely complex. Sometimes, the ducts are also wavy or corrugated in the streamwise direction and can be manufactured with ribs in order to achieve a faster transition to turbulence. It is well known that secondary motions take place in the smooth corner of non-circular straight ducts in the plane perpendicular to the streamwise direction. These motions, which are of Prandtl’s second kind, are turbulence induced. Such motions are important since they redistribute the kinetic energy, influence the axial velocity, and thereby affect the wall shear stress and heat transfer.

In the study reported here, a computational method is developed to predict the mean and turbulent flow field in arbitrary ducts using an explicit algebraic stress model. The method is applied to square, rectangular, trapezoidal, and triangular ducts with iso-thermal wall conditions using gradient-diffusion type heat flux models. The heat flux models are a simple eddy diffusivity model (SED), a generalized gradient diffusion hypothesis (GGDH) model, and a model extracted from the empirical WET hypothesis \cite{1}. The EASM representation is used for both high- and low-Reynolds numbers without introducing any damping functions into the tensor representation for the Reynolds stresses; however, at low Reynolds numbers, damping functions are introduced into the isotropic eddy viscosity and the heat flux models. At high Reynolds numbers ($\geq 10^5$), wall functions are used for both the velocity and temperature fields.
2. Mean and Turbulent Equations

A Reynolds-averaged Navier-Stokes (RANS) approach is used to predict the fully developed turbulent flow and heat transfer in the three-dimensional duct flow. The governing equations for the mean and turbulent quantities are

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j) = 0, \tag{1}
\]

\[
\frac{\partial \rho U_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}(-\rho U_i U_j)\right], \tag{2}
\]

\[
\frac{\partial (\rho T)}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j T) = \frac{\partial}{\partial x_j}\left[\mu \frac{\partial T}{\partial x_j}\right] + \frac{\partial}{\partial x_j}\left(-\rho U_j T\right). \tag{3}
\]

The turbulent stresses \(\rho \tau_{ij} = -\rho \bar{u}_i \bar{u}_j\) and turbulent heat fluxes \((\rho c_p u_i \bar{T})\) require modeling in order to close the equations.

For the modeling of the Reynolds stresses \(\rho \tau_{ij}\), within the context of an algebraic stress formulation, transport equations for the turbulent kinetic energy and turbulent dissipation rate are needed:

\[
\frac{\partial \rho \kappa}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j \kappa) = + \frac{\partial}{\partial x_j}\left[\left(\mu + \mu_t \right) \frac{\partial \kappa}{\partial x_j}\right] + P_k - \rho \varepsilon. \tag{4}
\]

\[
\frac{\partial \rho \varepsilon}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j \varepsilon) = + \frac{\partial}{\partial x_j}\left[\left(\mu + \mu_t \right) \frac{\partial \varepsilon}{\partial x_j}\right] + C_{\varepsilon 1} \frac{\varepsilon}{\kappa} P_k - f_2 C_{\varepsilon 2} \rho \frac{\varepsilon^2}{k}. \tag{5}
\]

where \(P_k = \tau_{ij} \partial U_i / \partial x_j\) is the production term. The constants \(C_{\varepsilon 1}\) and \(C_{\varepsilon 2}\) are set to 1.44 and 1.83, respectively, and the turbulent eddy viscosity is calculated as \(\mu_t = \rho f_\mu C_{\mu} \frac{k^2}{\varepsilon}\) where \(C_{\mu} \approx 0.09\). The functions \(f_1, f_2,\) and \(f_\mu\) are damping functions and are equal to unity in the fully turbulent region. In this study, the Abe-Kondoh-Nagano [2] formulations for \(f_1, f_2,\) and \(f_\mu\) are used, and are given by

\[
f_2 = \left(1 - e^{-\frac{k^2}{0.4 d}}\right)^2 \left[1 + 0.15 e^{-\frac{k^2}{0.4 d}}\right], \quad f_\mu = \left(1 - e^{-\frac{k^2}{0.4 d}}\right)^2 \left[1 + \frac{5}{(Re_t)^{0.75}} e^{-\frac{k^2}{0.4 d}}\right], \tag{6}
\]

where

\[
y^+ = \frac{\rho d}{\mu}, \quad Re_t = \frac{\rho \kappa^2}{\mu \varepsilon}, \tag{7}
\]

and \(d\) is the normal distance to the nearest wall. When the AKN model is used, the constants \(\sigma_k\) and \(\sigma_\varepsilon\) are both set to 1.4, and all the remaining constant coefficients are calibrated against DNS channel flow data [3]. At high Reynolds numbers, the coefficients \(\sigma_k = 1\) and \(\sigma_\varepsilon\) is determined from \(k^2/[\sqrt{C_{\mu} (C_{\varepsilon 2} - C_{\varepsilon 1})}].\)

The explicit algebraic stress model used is an extension of the Gatski and Speziale [4] model and described in Rumsey et al. [5]. In terms of \(\tau_{ij}\), it can be written as

\[
\tau_{ij} = \frac{2}{3} k \delta_{ij} + \alpha_1 S_{ij} + \alpha_2 (S_{ik} W_{kj} - W_{ik} S_{kj}) + \alpha_3 \left(S_{ik} S_{kj} - \frac{1}{3} \{S^2\} \delta_{ij}\right), \tag{8}
\]

where \(S_{ij}\) and \(W_{ij}\) are the mean strain rate and rotation rate tensors, respectively \((S_{ij} + W_{ij} = \partial U_i / \partial x_j)\). The \(\alpha_i\)’s are scalar coefficient functions of the invariants \(\eta^2 (= S_{ij} S_{ij} = \{S^2\})\), and \(\xi^2 (= W_{ij} W_{ij} = -\{W^2\})\), and are given by

\[
\alpha_3^{\text{a}} = \frac{\gamma_1}{\gamma_0 \eta^2 \tau^2} \alpha_1^{\text{a}} + \frac{1}{4 \gamma_0^4 \eta^4 \tau^2} \left[\gamma_1^{\text{a}} - 2a_1 \eta^2 \tau^2 - 2 \eta^2 \tau^2 \left(\frac{a_3^2}{3} - \mathcal{R}^2 a_2 \right)\right] \alpha_1 + \frac{a_1 \gamma_1}{4 \gamma_0^2 \eta^4 \tau} = 0, \tag{9}
\]

where \(\gamma_0 = 1.4, \gamma_1 = 1.4, \gamma_2 = 0.25\), and \(\mathcal{R} = \frac{\kappa}{\varepsilon} = \frac{1}{2} \frac{\kappa}{\varepsilon}\).

\(\gamma_0, \gamma_1, \gamma_2, \gamma_3\) are scalar coefficient functions of the invariants \(\eta^2 (= S_{ij} S_{ij} = \{S^2\})\), and \(\xi^2 (= W_{ij} W_{ij} = -\{W^2\})\), and are given by

\[
\gamma_3^{\text{a}} = \frac{\gamma_1}{\gamma_0 \eta^2 \tau^2} \gamma_2^{\text{a}} + \frac{1}{4 \gamma_0^4 \eta^4 \tau^2} \left[\gamma_2^{\text{a}} - 2a_1 \eta^2 \tau^2 - 2 \eta^2 \tau^2 \left(\frac{a_3^2}{3} - \mathcal{R}^2 a_2 \right)\right] \gamma_2 + \frac{a_1 \gamma_1}{4 \gamma_0^2 \eta^4 \tau} = 0. \tag{10}
\]
where
\[
\begin{align*}
\alpha_2 &= a_4 a_2 \alpha_1 \quad \text{and} \quad \alpha_3 = -2a_4 a_5 \alpha_1 \\
\end{align*}
\]

\[
\text{Eq. (10)}
\]

\[
\begin{align*}
a_1 &= 0.487, \quad a_2 = 0.80, \quad a_3 = 0.375, \quad a_4 = g \tau, \quad \tau = \frac{k}{\varepsilon}, \\
\end{align*}
\]

\[
\text{Eq. (11)}
\]

\[
\mathcal{R} = \frac{\xi^2}{\eta^2}, \quad g = \left[\gamma_1 - 2\gamma_0 \alpha_1 \eta^2 \tau\right]^{-1}, \quad \gamma_0 = 1.19 \text{ and } \gamma_1 = 0.7.
\]

\[
\text{Eq. (12)}
\]

The proper choice for \(\alpha_1\) is the minimum real root of Eq. (9) [6].

Three different models are used to provide closure for the turbulent heat flux term. The first is an isotropic, simple eddy diffusivity (SED) model based on the Boussinesq approximation,

\[
\overline{u_j T} = -\frac{\mu_t}{\rho \sigma_T} \frac{\partial T}{\partial x_j}
\]

where the turbulent Prandtl-number for temperature \(\sigma_T\) is set to 0.89. The second is a model based on a generalized gradient diffusion hypothesis (GGDH),

\[
\overline{u_j T} = -C_t \frac{k}{\varepsilon} \left(\frac{\partial T}{\partial x_k}\right)
\]

and the third model is based on the WET method and is given by

\[
\overline{u_j T} = -C_t \frac{k}{\varepsilon} \left(\frac{\partial T}{\partial x_k} + \frac{\partial U_j}{\partial x_k}\right),
\]

where \(C_t = 0.3\) in both the GGDH and the WET models. At low-Reynolds numbers, a damping function for the SED model \(f_\mu\) is included in the turbulent eddy viscosity \(\mu_t\), and for the GGDH and WET models, a damping function \(f_\mu T\) is introduced. This damping function is a Lam-Bremhorst (LB) type model which is given by

\[
f_\mu T = \left(1 - e^{-0.0225 R_k}\right)^2 \left(1 + \frac{41}{Re_t}\right),
\]

where \(Re_k = \rho \sqrt{k d}/\mu\). If the same number of grid points is used in the cross-section, it was found that using the LB model in the GGDH and WET closures gave better results than the AKN model for flows where \(Re > 10^4\). Note that the WET model is implicit and the resulting system of equations for the heat fluxes are solved analytically in each iteration (no numerical inner iteration loop).

Both Fanning friction factor and Nusselt number have been obtained from the computations. The calculated friction factor is thus related to the Prandtl-law [7] as

\[
\frac{1}{\sqrt{4 f}} = 2 \log \left(Re \sqrt{4 f}\right) - 0.8.
\]

The \(Re\) number is based on the hydraulic diameter defined for two or three walls as

\[
D_h = \frac{4 A_{\text{cross}}}{a + \frac{b}{\sin \phi}} \quad \text{or} \quad D_h = \frac{4 A_{\text{cross}}}{a + b + \frac{b}{\sin \phi}},
\]

\[
\text{Eq. (18)}
\]
where \( a, b, h, \) and \( \phi \) are base-length, upper length, height and base-angle, respectively. The reference to two or three walls is to the number of walls in the cross-section when symmetry conditions are used, and \( A_{\text{cross}} \) is the cross-section area which can be defined as \( 0.5(a + h) \) for all cases considered here.

The calculated \( Nu \)-number is related to the Dittus-Boelter correlation [7] by

\[
Nu = 0.023 Re^{0.8} Pr^{0.3} \quad \text{for} \quad Re \geq 8000
\]  

(19)

At high Reynolds numbers, the law of the wall is assumed to be valid for both the velocity and temperature fields in the near wall region. A further discussion of the specification and implementation of the boundary conditions, as well as the numerical procedure used in the solution of the mean and turbulent equations can be found in Rokni and Sundén [8, 9].

3. Results

Straight ducts with square, rectangular, trapezoidal, and triangular cross-sections are considered in this investigation. Only one quarter of the duct with a square (rectangular) cross-section, and only half of the duct with trapezoidal and triangular cross-section are considered by imposing symmetry conditions. Sketches of the various grid structures within the class of duct configurations under consideration here are shown in Fig. 1. The calculations focus on fully developed, three-dimensional, turbulent flow. Results of mean velocity and temperature, and friction factor and \( Nu \)-number distributions are presented. The latter two quantities being the most important hydraulic parameters from an engineering standpoint. In addition, the secondary flow generated within the ducts is also analyzed.

3.1. Square Ducts

While the square duct flow field has been the subject of many previous studies, it can be used here to illustrate the range of applicability and robustness of the current methodology. As alluded to previously, the present calculation procedure is highly stable and can be extended to much higher \( Re \) number than \( 10^4 \) with a minimal demand on the number of grid points. In Fig. 2 all the calculations were performed with only \( 31 \times 31 \) grid points for all Reynolds numbers. No convergence problems arose even at this high \( Re \) number by using the present models. For
the hydraulic parameter such as Fanning friction factor and Nusselt number the results are quite good. Figure 2 shows that the friction factor obtained from the EASM is over-predicted by about 5% compared to the Prandtl-law correlation while the $Nu$-number predicted by GGDH and WET agree very well with the Dittus-Boelter correlation.

3.2. Rectangular Ducts

Different rectangular ducts (side ratio 2,3,5 and 10) are considered. Even in these cases, if the GGDH and WET models are used, both the friction factor and $Nu$-number predicted by the EASM agree very well with the theoretical correlations. If the SED model is used, the $Nu$-number is under-predicted by about 15%. In Fig. 3, the secondary flow motion for two rectangular ducts with aspect ratio 3 and 5 are shown. The secondary motion velocity vectors predicted by the EASM, with the AKN damping functions in the $k - \varepsilon$ equations, are in good agreement with what has been observed in some experimental results. The existence of such secondary flow patterns was first observed by Nikuradse during his experiment with non-circular ducts (see [10]).

Table 1 provides calculated Fanning friction factor, $Nu$-number and the center to bulk-velocity ratio ($U_c/U_b$) in a rectangular duct with different aspect ratios. For a given cross section the $U_c/U_b$ decreases slightly with increasing $Re$ number, which is also evident from this table. The experimental value of $U_c/U_b$ for a rectangular duct with aspect ratio 8 at $Re \approx 5800$ is 1.23 (see [11]) which can be compared with the calculated result (Table 1) of 1.22 for an aspect ratio of 10 at $Re \approx 1.572 \times 10^3$.

3.3. Trapezoidal and Triangular Ducts

The velocity vectors and the corresponding mean flow contours predicted by the EASM in a trapezoidal duct are presented in Fig. 4. As shown in the figure, there exist two counter rotating vortices close to each corner which are similar to the results obtained by Rokni and Sundén [8]. Only $61 \times 31$ grid points were used in the cross-section to perform the calculation. Since the LB damping functions had convergence and stability problems regardless of grid arrangement in the cross-section in the trapezoidal ducts, the AKN damping functions were used. In Fig. 4,
Figure 3: Predicted secondary motion velocity vectors in rectangular ducts with aspect ratio 3 and 5 using EASM with damping functions.

Table 1: Calculated $Nu$-number and Fanning friction factor for rectangular ducts with different aspect ratios using EASM and GGDH.

<table>
<thead>
<tr>
<th>Aspect Ratio</th>
<th>$Re \times 10^4$</th>
<th>$f \times 10^3$</th>
<th>$Nu$</th>
<th>$U_c/U_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.9397</td>
<td>8.165</td>
<td>30.4</td>
<td>1.28</td>
</tr>
<tr>
<td>3</td>
<td>1.1474</td>
<td>7.797</td>
<td>35.8</td>
<td>1.28</td>
</tr>
<tr>
<td>5</td>
<td>1.3666</td>
<td>7.541</td>
<td>41.2</td>
<td>1.27</td>
</tr>
<tr>
<td>10</td>
<td>1.5717</td>
<td>7.401</td>
<td>47.3</td>
<td>1.22</td>
</tr>
</tbody>
</table>

$Re \approx 1.546 \times 10^4$ and the calculated Fanning friction factor and $Nu$-number (GGDH model) are $7.791 \times 10^{-3}$ and 48.1, respectively. These values can be compared with the Prandtl-law and Dittus-Boelter correlations, Eqs. (17) and (19), which yield $6.900 \times 10^{-3}$ and 46.8 respectively. The center to bulk-velocity ratio ($U_c/U_b$) is calculated as 1.29.

Close to the upper side corner ("north wall" and "high wall") there exist two counter rotating vortices - a small one and a much larger one. The smaller vortex size decreases with decreasing upper side length ("north wall") until it vanishes for a triangular duct. Correspondingly, the large vortex size increases while this length decreases (see Fig. 4). This type of secondary flow pattern in a triangular duct was also observed in the experiment of Nikuradse (see [10]).

The highly stable nature of the calculation procedure used here makes it possible to apply the present models to triangular ducts and to predict turbulence quantities without any convergence problems. In Fig. 5 the upper side length is much smaller than the two other lengths ($\approx 2 \times 10^{-3}$ of the duct height). This length cannot be set to zero since using structured grids in the calculations requires that no side of any control volume in the domain be zero. Nevertheless, this very small upper side length would still yield the correct limiting behavior of a sharp corner,
and would be a case where many turbulence models would fail. The $Re$ number in this duct is $1.164 \times 10^4$ and the predicted Fanning friction factor and $Nu$-number (GGDH model) are $8.016 \times 10^{-3}$ and 36.3, respectively. These results can be compared with the Prandtl-law and Dittus-Boelter correlations (Eqs. 17 and 19) which yield $7.421 \times 10^{-3}$ and 37.3, respectively. The center to bulk-velocity ratio ($U_c/U_b$) is calculated as 1.30.

Figure 5: Predicted velocity and temperature fields in a triangular duct: (a) secondary motion velocity vectors; (b) mean temperature contours.

4. Summary

The results from the numerical solution of fully developed, three-dimensional turbulent duct flow under iso-thermal conditions have been presented for square, rectangular, trapezoidal and triangular ducts. The turbulent stresses were modeled using an EASM, and the turbulent heat
fluxes were modeled by the SED, GGDH and WET methods. At high Reynolds numbers ($\geq 10^5$), wall functions for the velocity and temperature fields were used. At low Reynolds numbers, the AKN damping functions were used for the turbulent equations, and for the turbulent heat fluxes (GGDH and WET methods) Lam-Bremhorst type damping functions were used. Excellent results were obtained in term of friction factor, $N_u$-number, and secondary motion, and with minimal computational cost.

These results suggest that while the models for the heat fluxes can be very simple, this does not necessarily preclude an accurate prediction of the temperature field. Under iso-thermal conditions, simple gradient-diffusion models for the heat fluxes may suffice if the flow field can be well predicted. In complex geometries such as those examined here, it is necessary to use higher-order models for the Reynolds stresses, since anisotropies in the turbulent stress field are important and simple linear eddy viscosity models will not suffice. Higher-order closures for the heat fluxes may also be required in non-adiabatic cases and/or in cases where counter-gradient heat transfer occurs. Corresponding explicit algebraic heat flux models, coupled with equations for the temperature variance and variance dissipation rate, could be applied to such flows.

References