Aspects of the Tutte polynomial

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This thesis studies various aspects of the Tutte polynomial, especially focusing on the Merino-Welsh conjecture.

We write \( T(G;x,y) \) for the Tutte polynomial of a graph \( G \) with variables \( x \) and \( y \). In 1999, Merino and Welsh conjectured that if \( G \) is a loopless 2-connected graph, then

\[
T(G;1,1) \leq \max\{T(G;2,0), T(G;0,2)\}.
\]

The three numbers, \( T(G;1,1) \), \( T(G;2,0) \) and \( T(G;0,2) \) are respectively the numbers of spanning trees, acyclic orientations and totally cyclic orientations of \( G \).

First, I extend Negami's splitting formula to the multivariate Tutte polynomial. Using the splitting formula, Thomassen and I found a lower bound for the number of spanning trees in a \( k \)-edge-connected graph. Our bound is tight for \( k \) even, but for \( k \) odd we give a slightly better lower bound which we believe is not tight. We prove that the minimum number of spanning trees in a 3-edge-connected graph with \( n \) vertices is, not surprisingly, significantly smaller than the minimum number of spanning trees in a 4-edge-connected graph. However, we conjecture that the minimum number of spanning trees of a 5-edge-connected graph is actually obtained by a 6-edge-connected graph asymptotically.

Thomassen proved the following partial result for the Merino-Welsh conjecture. Assume the graph \( G \) is loopless, bridgeless and has \( n \) vertices and \( m \) edges.

- If \( m \leq 1.066 n \) then \( T(G;1,1) \leq T(G;2,0) \).
- If \( m \geq 3.58(n-1) \) and \( G \) is 3-edge-connected then \( T(G;1,1) \leq T(G;0,2) \).

I improve in this thesis Thomassen's result as follows:

- If \( m \leq 1.29(n-1) \) then \( T(G;1,1) \leq T(G;2,0) \).
- If \( m \geq 3.58(n-1) \) and \( G \) is 3-edge-connected then \( T(G;1,1) \leq T(G;0,2) \).

Strengthening Thomassen's idea that acyclic orientations dominate spanning trees in sparse graphs, I conjecture that the ratio \( T(G;2,0)/T(G;1,1) \) increases as \( G \) gets sparser. To support this conjecture, I prove a variant of the conjecture for series-parallel graphs.

The Merino-Welsh conjecture has a stronger version claiming that the Tutte polynomial is convex on the line segment between \((2,0)\) and \((0,2)\) for loopless 2-connected graphs. Chavez-Lomeli et al. proved that this holds for coloopless paving matroids, and I provide a shorter proof of their theorem. I also prove it for minimally 2-edge-connected graphs. As a general statement for the convexity of the Tutte polynomials, I show that the Tutte polynomial of a sparse paving matroid with fixed rank is almost never convex in the first quadrant. In contrast, I conjecture that the Tutte polynomial of a sparse paving matroid with fixed rank is almost never convex in the first quadrant.

The following multiplicative version of the Merino-Welsh conjecture was considered by Noble and Royle:

\[
T(G;1,1)^2 \leq T(G;2,0) T(G;0,2).
\]

Noble and Royle proved that this multiplicative version holds for series-parallel graphs, using a computer algorithm that they designed. Using a property of the splitting formula which I found, I improve their algorithm so that it is applicable to the class of graphs with bounded treewidth (or pathwidth). As an application, I verify that the multiplicative version holds for graphs with pathwidth at most 3.

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